Lecture 5

Dynamics of the Universe

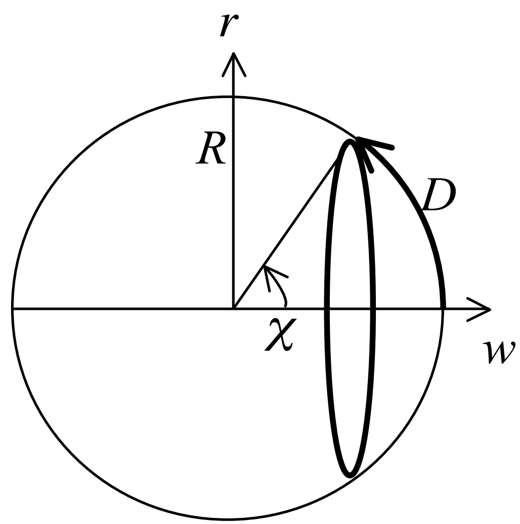
$R(t) = ?$

Robertson-Walker metric
uniformly curved, evolving spacetime

$$ds^2 = -c^2 dt^2 + R^2(t)\left(d\chi^2 + S_k^2(\chi)\, d\psi^2\right)$$

$$= -c^2 dt^2 + R^2(t)\left(\frac{du^2}{1 - k\, u^2} + u^2\, d\psi^2\right)$$

$$= -c^2 dt^2 + a^2(t)\left(\frac{dr^2}{1 - k\left(r/R_0\right)^2} + r^2\, d\psi^2\right)$$



$$S_k(\chi) = \begin{cases} \sin\chi & (k = +1) \quad \text{closed} \\ \chi & (k = 0) \quad \text{flat} \\ \sinh\chi & (k = -1) \quad \text{open} \end{cases}$$

$$d\psi^2 \equiv d\theta^2 + \sin^2\theta\, d\phi^2$$

$$a(t) \equiv R(t)/R_0$$

$$R_0 \equiv R(t_0)$$

radial distance $= D(t) = R(t)\,\chi$

circumference $= 2\pi\, r(t)$ $\qquad r(t) = a(t)\, r = R(t)\, u = R(t)\, S_k\left(\chi\right)$

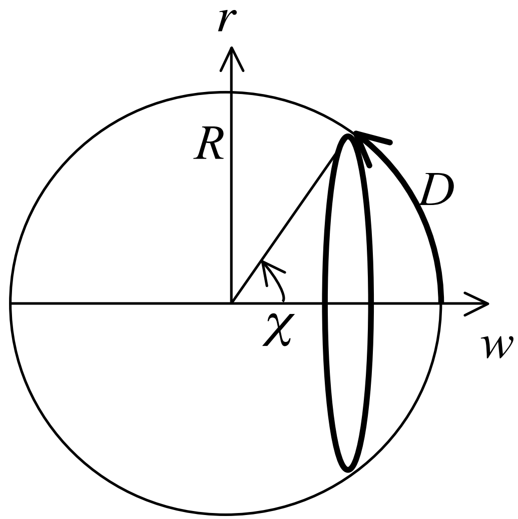
Time and Distance vs Redshift

- We observe the **redshift :** $z \equiv \frac{\lambda - \lambda_0}{\lambda_0} = \frac{\lambda}{\lambda_0} - 1 \quad \lambda = \text{observed},$

  $\lambda_0 = \text{emitted (rest)}$

- Hence we know the **expansion factor:**

$$x \equiv 1 + z = \frac{\lambda}{\lambda_0} = \frac{\lambda(t_0)}{\lambda(t)} = \frac{R(t_0)}{R(t)} = \frac{R_0}{R(t)}$$



- When was the light emitted? $t(z) = ?$
- How far away was the source? $\chi(z) = ?$

$$D(t,\chi) = R(t)\,\chi \qquad D_A = r_0(\chi)/\left(1+z\right)$$

$$r(t,\chi) = R(t)\,S_k(\chi) \qquad D_L = r_0(\chi)\left(1+z\right)$$

- How do these depend on cosmological parameters?

$$H_0 \quad \Omega_M \quad \Omega_\Lambda$$

Time -- Redshift relation

$$x = 1 + z = \frac{R_0}{R}$$

Memorise this derivation!

$$\frac{dx}{dt} = -\frac{R_0}{R^2}\frac{dR}{dt}$$

$$= -\frac{R_0}{R}\frac{\dot{R}}{R}$$

Hubble parameter : $H \equiv \frac{\dot{R}}{R}$

$$= -x\, H(x)$$

$$\therefore \boxed{dt = \frac{-dx}{x\, H(x)} = \frac{-dz}{(1+z)\, H(z)}}$$

Time and Distance vs Redshift

$$\frac{d}{dt}\left(x = 1 + z = \frac{R_0}{R} \right) \quad \rightarrow \quad dt = \frac{-dx}{x\,H(x)}$$

Look - back time :

$$t(z) = \int_t^{t_0} dt = \int_{1+z}^{1} \frac{-dx}{x\,H(x)} = \int_1^{1+z} \frac{dx}{x\,H(x)}$$

Age : $\quad t_0 = t(z \rightarrow \infty)$

Distance : $\quad D = R\,\chi \qquad r = R\,S_k(\chi)$

$$\chi(z) = \int d\chi = \int_t^{t_0} \frac{c\,dt}{R(t)} = \frac{c}{R_0} \int_1^{1+z} \frac{R_0}{R(t)} \frac{dx}{x\,H(x)} = \frac{c}{R_0} \int_1^{1+z} \frac{dx}{H(x)}$$

Horizon : $\quad \chi_H = \chi(z \rightarrow \infty)$

Need to know *R(t)*, or *R_0* and *H(x)*.

Einstein's General Relativity

- 1. ***Spacetime geometry tells matter how to move***
  - gravity = effect of curved spacetime
  - free particles follow geodesic trajectories
    - $ds^2 < 0$ v < c time-like massive particles
    - $ds^2 = 0$ v = c null massless particles (photons)
    - $ds^2 > 0$ v > c space-like tachyons (not observed)

- 2. ***Matter (+energy) tells spacetime how to curve***
  - Einstein field equations
    - nonlinear
    - second-order derivatives of metric with respect to space/time coordinates

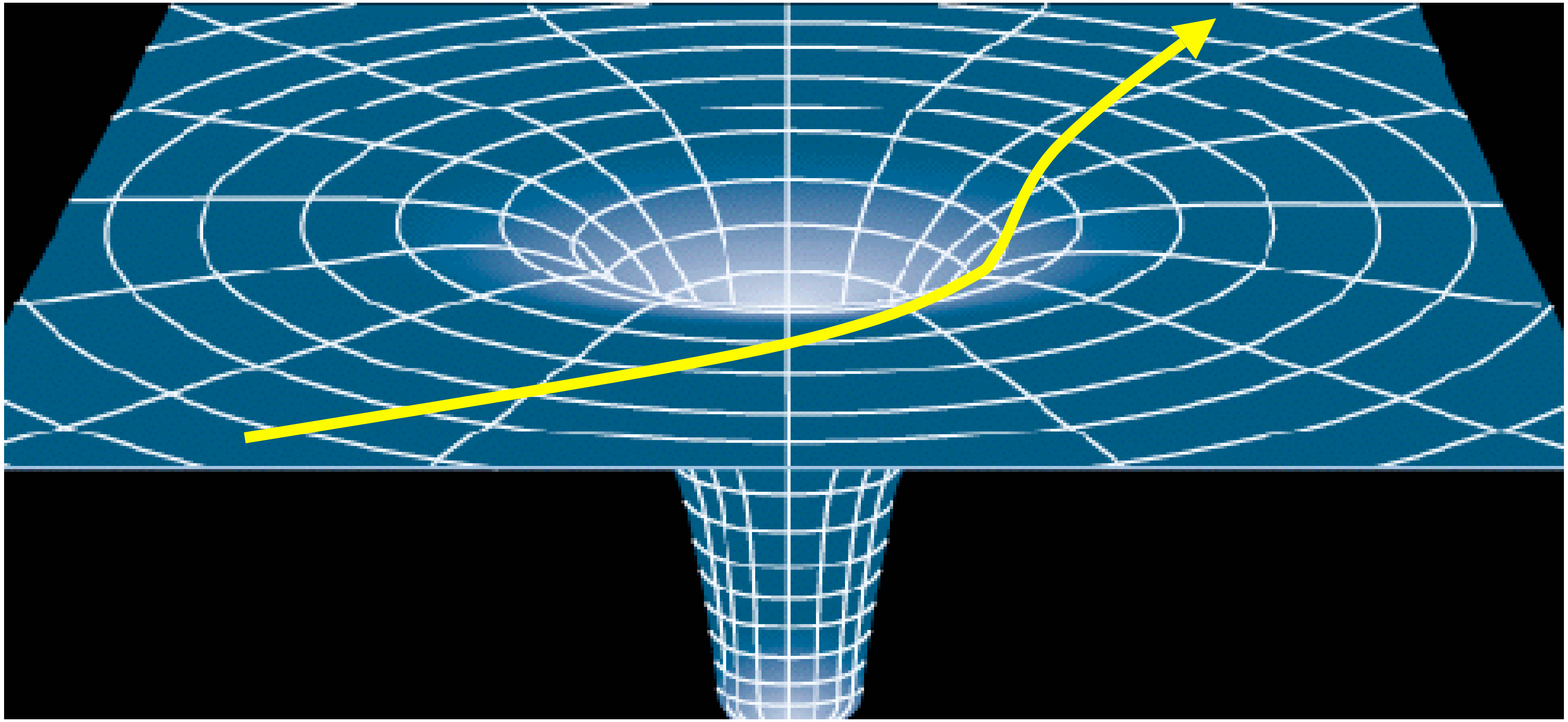
Geodesics

Gravity = curvature of space-time by matter/energy.

Freely-falling bodies follow **geodesic trajectories**.

Shortest possible path in curved space-time.

Local curvature replaces forces acting at distance.



Space-Time Geodesics

Schwarzschild Metric:

(curved space-time outside a compact mass *M*)

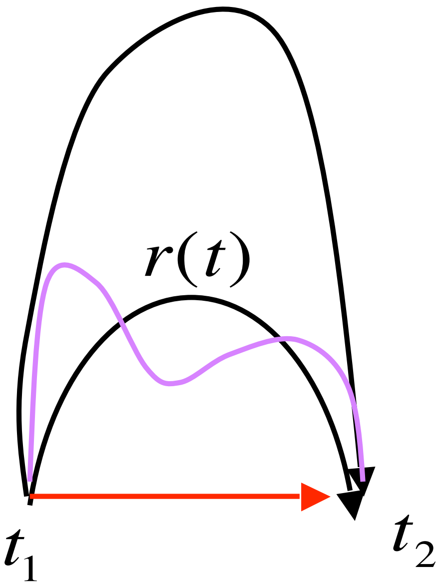
$$ds^2 = -c^2 d\tau^2 = -\left(1 - \frac{r_S}{r}\right) c^2 dt^2 + \frac{dr^2}{\left(1 - \frac{r_S}{r}\right)} + r^2 d\psi^2$$

Schwarzschild Radius:

$$r_S = \frac{2GM}{c^2}$$

Freely-falling test particles follow a ***geodessic path.***

Free fall = maximum τ proper time.



$$\tau = \int d\tau = \int_{t_1}^{t_2} \frac{d\tau}{dt}\, dt$$

$$d\tau^2 = \frac{-ds^2}{c^2} = \left(1 - \frac{r_S}{r}\right) dt^2 - \frac{1}{c^2}\left[\frac{dr^2}{\left(1 - \frac{r_S}{r}\right)} + r^2 d\varphi^2\right]$$

Space-Time Geodesics

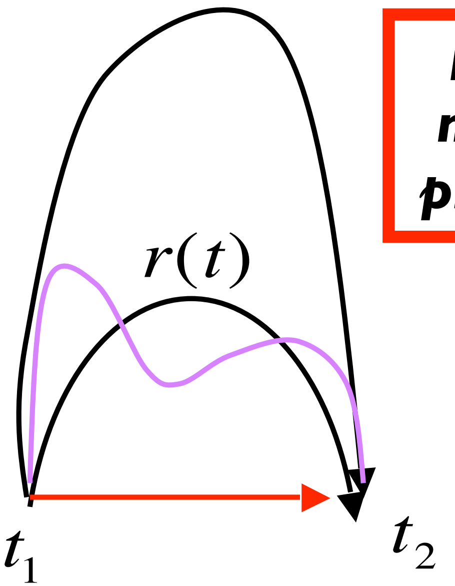
Schwarzschild Metric:

$$ds^2 = -c^2 d\tau^2 = -\left[1 - \left(r_S / r\right)\right] c^2 dt^2 + \frac{dr^2}{\left[1 - \left(r_S / r\right)\right]} \qquad r_S = \frac{2GM}{c^2}$$

Proper time:

High up clock runs faster :)

$$\frac{d\tau}{dt} = \sqrt{\frac{-ds^2}{c^2 dt^2}} = \frac{1}{c}\sqrt{-g_{tt}} = \sqrt{1 - \frac{2GM}{c^2 r}}$$



Free fall = maximum τ proper time.

Moving clock runs slower :(

$$\frac{d\tau}{dt} = \frac{1}{c}\sqrt{-g_{tt} - g_{rr}\left(\frac{dr}{dt}\right)^2} = \sqrt{1 - \frac{2GM}{c^2 r} - \frac{v^2}{c^2} + \ldots}$$

$$mc^2\tau = \int_{t_1}^{t_2}\left(mc^2 - \frac{GMm}{r} - \frac{mv^2}{2} + \ldots\right)dt \approx (t_2 - t_1)mc^2 - \int L(t)\ dt$$

$$\max[\tau] = \min[\int L\, dt] \qquad L \equiv E_{kin} - E_{pot} = \text{Lagrangian}$$

Tests of General Relativity

- Matter bends light rays (e.g. solar eclipses)
  - 1.75 arcsec for Sun grazing ray
- Perihelion shift of planetary orbits

	GR	observed	
Mercury	43.03	43.11 +/- 0.45	arcsec/century
Venus	8.6	8.4 +/- 4.8	
Earth	3.8	5.0 +/- 1.2	
Icarus	10.3	9.8 +/- 0.8	

- Time delay of Venus radar reflections
  - ~200 microsec
- Decay of orbit of binary pulsars
  - due to emission of gravitational waves
- All tests on small size scales.
- For cosmology, assume GR valid on larger scales.