Lecture 4

Space-Time Metric
**Olber’s Paradox**

*Why is the sky dark at night?*

Flux from all stars in the sky:

\[
F = \int n_* F_* \, d(Vol) = \int_0^{\chi_{\text{max}}} n_* \left( \frac{L_*}{A(\chi)} \right) (A(\chi) R \, d\chi)
\]

\[
= n_* L_* R \, \chi_{\text{max}}
\]

\[
\Rightarrow \infty \quad \text{for flat space, } R \rightarrow \infty.
\]

A dark sky may imply:

1. an edge (we don't observe one)
2. a curved space (finite size)
3. expansion \((R(t) \Rightarrow \text{finite age, redshift})\)
Minkowski Spacetime Metric

\[ ds^2 = -c^2 dt^2 + dl^2 \]

\[ d\tau^2 = dt^2 - \frac{dl^2}{c^2} = dt^2 \left( 1 - \frac{1}{c^2} \left( \frac{dl}{dt} \right)^2 \right) \]

**Proper time (moving clock):**

\[ d\tau = \sqrt{-ds^2/c^2} = dt \sqrt{1 - \frac{v^2}{c^2}} > 0 \]

**Time-like intervals:**

\[ ds^2 < 0, \quad d\tau^2 > 0 \]

Inside light cone. Causally connected.

**Space-like intervals:**

\[ ds^2 > 0, \quad d\tau^2 < 0 \]

Outside light cone. Causally disconnected.

**Null intervals light cone:**

\[ v = c, \quad ds^2 = 0 \]

Photons arrive from our past light cone.

World line of massive particle at rest.

Null intervals light cone:
**Minkowski Spacetime Metric**

\[ ds^2 = -c^2 dt^2 + dl^2 \]

\[ d\tau^2 = dt^2 - \frac{dl^2}{c^2} = dt^2 \left( 1 - \frac{1}{c^2} \left( \frac{dl}{dt} \right)^2 \right) \]

**Time-like intervals:** \( ds^2 < 0, \ d\tau^2 > 0 \)

Inside light cone. Causally connected.

**Space-like intervals:** \( ds^2 > 0, \ d\tau^2 < 0 \)

Outside light cone. Causally disconnected.

**Null intervals light cone:** \( v = c, \ ds^2 = 0 \)

Photons arrive from our past light cone.

**Proper time (moving clock):**

\[ d\tau = \sqrt{-ds^2/c^2} \]

\[ = dt \sqrt{1 - \frac{v^2}{c^2}} > 0 \]

World line of massive particle in motion.
**Robertson-Walker metric**

uniformly curved, evolving spacetime

\[
 ds^2 = -c^2 dt^2 + R^2(t) \left( d\chi^2 + S_k(\chi) d\psi^2 \right) \\
 = -c^2 dt^2 + R^2(t) \left( \frac{du^2}{1 - k u^2} + u^2 d\psi^2 \right) \\
 = -c^2 dt^2 + a^2(t) \left( \frac{dr^2}{1 - k (r/R_0)^2} + r^2 d\psi^2 \right)
\]

\[
 S_k(\chi) = \begin{cases} 
 \sin \chi & (k = +1) \quad \text{closed} \\
 \chi & (k = 0) \quad \text{flat} \\
 \sinh \chi & (k = -1) \quad \text{open}
\end{cases}
\]

\[
 d\psi^2 \equiv d\theta^2 + \sin^2 \theta \, d\phi^2
\]

radial distance \( = D(t) = R(t) \chi \)

circumference \( = 2\pi r(t) \quad r(t) = a(t) \quad r = R(t) \quad u = R(t) S_k(\chi) \)
Redshift and Time Dilation

Light rays are null geodessics:

\[ ds^2 = R^2(t) \, d\chi^2 - c^2 \, dt^2 = 0 \]

\[ d\chi = \frac{c \, dt}{R(t)} \]

\[
\chi = \int_{t_e}^{t_e + \Delta t_e} \frac{c \, dt}{R(t)} + \int_{t_e + \Delta t_e}^{t_o} \frac{c \, dt}{R(t)} = \int_{t_e}^{t_o} \frac{c \, dt}{R(t)} + \int_{t_o}^{t_o + \Delta t_o} \frac{c \, dt}{R(t)}
\]

\[
\frac{\Delta t_e}{R(t_e)} = \frac{\Delta t_o}{R(t_o)} \quad \Rightarrow \quad \frac{R(t_o)}{R(t_e)} = \frac{\Delta t_o}{\Delta t_e} = \frac{\lambda_o}{\lambda_e} = 1 + z
\]

\[
\therefore \text{Observed wavelengths and time intervals appear "stretched" by a factor } x = 1 + z = \frac{R_0}{R(t)}.
\]
Fidos and co-moving coordinates

Distance varies in time:

\[ D(t) \]

“Fiducial observers” (Fidos)

\[ D(t) = R(t) \chi \]

“Co-moving” coordinates

\[ \chi \] or \[ D_0 \equiv R_0 \chi \]

Labels the Fidos
Cosmological Principle

All “co-moving” observers see an equivalent view.

Past light cone:
- looking out = back in time
Coordinate Systems
Conformal time

Stretch time axis to make light rays at 45°.

\[ \eta \equiv \int \frac{c \, dt}{R(t)} \]

\[ ds^2 = R(t)^2 \left( -d\eta^2 + d\chi^2 + S_k^2(\chi) \, d\psi^2 \right) \]
The Horizon: How far can we see?

- We see only a finite patch of the Universe.
- The Horizon may grow with time (if expansion decelerates) or shrink in time (e.g. inflation).

\[ R(t) = R_0 \left( \frac{t}{t_0} \right)^\alpha \]

\[ \chi_H = \eta_0 = \int_0^{t_0} \frac{c \, dt}{R(t)} = \frac{c \, t_0}{(1 - \alpha) \, R_0} \]

if \( \alpha < 1 \)

\[ \Rightarrow \infty \]

as \( \alpha \to 1 \)

\( \alpha = 2/3 \) : matter-dominated

\( \alpha = 1/2 \) : radiation-dominated
Angular Diameter Distance

- radial distance
  - now (when photon received):
    \[ D_0 = R(t_0) \chi = R_0 \chi \]
  - when photon emitted:
    \[ D_e = R(t_e) \chi = \frac{R(t_e) R_0 \chi}{R_0} = \frac{D_0}{1+z} \]

- angular size
  - Fraction of circumference when photon was emitted:
    \[ \frac{\theta}{2\pi} = \frac{l}{2\pi r(t_e)} \]

- angular diameter distance
  \[ D_A = \frac{l}{\theta} = r(t_e) = R(t_e) S_k(\chi) = \frac{R(t_e)}{R_0} R_0 S_k(\chi) \]
  \[ = \frac{R_0 S_k(\chi)}{1+z} = \frac{r(t_0)}{1+z} \equiv \frac{r_0}{1+z} \]

Circumference was smaller by factor \(x=1+z\).
Sources look larger/closer.
Luminosity Distance

- Luminosity (erg s\(^{-1}\))
  \[ L = \frac{N \ h \ \nu_e}{\Delta t_e} \]
- Area of photon sphere (when photons observed):
  \[ A_0 = 4\pi r_0^2 = 4\pi R_0^2 S_k^2(\chi) \]
- Redshift:
  \[ \lambda_0 = \lambda_e (1 + z) \]
  \[ \nu_0 = \nu_e / (1 + z) \]
- Time dilation: lower photon arrival rate
  \[ \Delta t_0 = \Delta t_e (1 + z) \]
- Observed flux (erg cm\(^{-2}\) s\(^{-1}\))
  \[ F = \frac{N \ h \ \nu_0}{A_0 \Delta t_0} = \frac{L}{4\pi r_0^2 (1 + z)^2} = \frac{L}{4\pi D_L^2} \]

- Luminosity distance
  \[ D_L = (1 + z) r_0 = (1 + z) R_0 S_k(\chi) \]

Sources look fainter/farther.
Surface Brightness

- **Solid angle**
  \[ \Omega = \frac{A}{D_A^2} \]

- **Surface brightness**
  - Flux per solid angle (erg s\(^{-1}\) cm\(^{-2}\) arcsec\(^{-2}\))
  \[ \Sigma \equiv \frac{F}{\Omega} = \frac{L}{4\pi D_L^2} \frac{D_A^2}{A} = \frac{L}{4\pi A (1+z)^4} \]
  - decreases very rapidly with z because:
  - expansion spreads out the photons
  - decreases their energy
  - decreases their arrival rate

- \( D_L = (1+z) r_0 \)
- \( D_A = \frac{r_0}{(1+z)} \)
- \( r_0 = R_0 S_k(\chi) \)
**Flux Density Spectra**

- **emitted photons**
  - \( L_{\nu}(\lambda_e) = \frac{N(\lambda_e) h \nu_e}{\Delta \nu_e \Delta t_e} \) (erg s\(^{-1}\) Hz\(^{-1}\))

- **redshift**
  - \( \lambda_o = \lambda_e (1 + z) \)

- **observed flux density spectra**
  - \( F_{\nu}(\lambda_o) = \frac{N(\lambda_o) h \nu_0}{\Delta \nu_0 \Delta t_0 A} = \frac{L_{\nu}(\lambda_e)}{4 \pi D_A^2} \frac{1}{(1 + z)} \) (erg cm\(^{-2}\) s\(^{-1}\) Hz\(^{-1}\))
  - \( F_{\lambda}(\lambda_o) = \frac{N(\lambda_o) h \nu_0}{\Delta \lambda_0 \Delta t_0 A} = \frac{L_{\lambda}(\lambda_e)}{4 \pi D_A^2} \frac{1}{(1 + z)^3} \) (erg cm\(^{-2}\) s\(^{-1}\) Å\(^{-1}\))