Radial Potential

energy and angular momentum:

$$e \equiv \frac{E}{m} = \frac{1}{2}\left[\dot{r}^2 + r^2\dot{q}^2\right] - \frac{G M}{r}$$

$$L \equiv \frac{J}{m} = r^2 \dot{q}$$

effective potential: $e = \frac{\dot{r}^2}{2} + \Phi(r)$

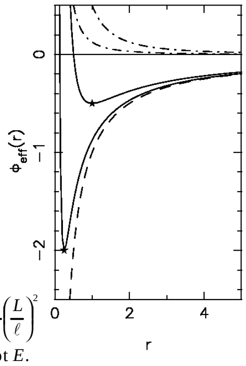
$$\Phi(r) = \frac{L^2}{2 r^2} - \frac{G M}{r} = e_o\left(\frac{\ell^2}{r^2} - 2\frac{\ell}{r}\right)$$

circular orbit:

$$\frac{\partial \Phi}{\partial r} = \frac{G M}{r^2} - \frac{L^2}{r^3} = 0 \rightarrow r = \frac{L^2}{G M} \equiv \ell$$

$$\Phi(\ell) = -e_o \quad e_o \equiv \frac{G M}{2\,\ell} = \frac{1}{2}\left(\frac{G M}{L}\right)^2 = \frac{1}{2}\left(\frac{L}{\ell}\right)^2$$

Note: ℓ, e_o depend on M and J but not E.



Types of Orbits

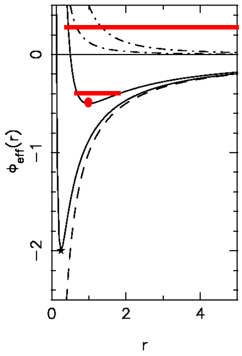
Fix L, E = min circular

$E < 0$ bound (ellipse)

$E > 0$ unbound (hyperbola)

Fix $E < 0$, L = max circular

L = min radial



Turning Points

energy and angular momentum :

$$e = \frac{\dot{r}^2}{2} + \frac{L^2}{2\,r^2} - \frac{G M}{r}$$

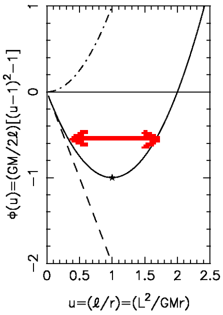
$$= \frac{\dot{r}^2}{2} + e_o\left[\left(u-1\right)^2 - 1\right]$$

$$u \equiv \frac{\ell}{r}$$

turning points:

$$\dot{r}^2 = 0 \rightarrow \left(u-1\right)^2 = 1 + \frac{e}{e_o} \equiv e^2$$

$$u^{\pm} = \frac{\ell}{r^{\mp}} = 1 \pm \left(1 + \frac{e}{e_0}\right)^{1/2}$$



Orbit Shape

$$e = \frac{\dot{r}^2}{2} + e_0\left(x^2 - 1\right)$$

$$\dot{r}^2 = 2\,e_0\left(\frac{e}{e_0} + 1 - x^2\right) = \left(\frac{L}{\ell}\right)^2\left(e^2 - x^2\right)$$

$$\frac{d\,x}{d q} = \frac{\dot{x}}{\dot{q}} = \left(-\frac{\ell\,\dot{r}}{r^2}\right)\Big/\left(\frac{L}{r^2}\right) = -\frac{\ell}{L}\,\dot{r}$$

$$\left(\frac{d\,x}{d q}\right)^2 + x^2 = e^2$$

simple harmonic oscillator !

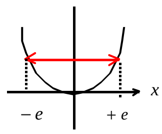
$$x = e\cos q \quad \rightarrow \quad \boxed{r = \frac{\ell}{1 + e\cos q}}$$

$$x \equiv \frac{\ell}{r} - 1 \qquad \dot{x} = -\frac{\ell\,\dot{r}}{r^2}$$

$$\ell \equiv \frac{L^2}{G\,M} \qquad L \equiv r^2\dot{q}$$

$$e_0 \equiv \frac{1}{2}\left(\frac{L}{\ell}\right)^2$$

$$e^2 \equiv \frac{e}{e_0} + 1$$



Conic Sections

$$r = \frac{\ell}{1 + e\cos q}$$

e = eccentricity

> 1 hyperbola

= 1 parabola

< 1 ellipse

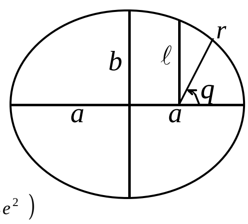
= 0 circle

ℓ = semi-latus rectum $= a\left(1 - e^2\right)$

a = semi-major axis

b = semi-minor axis

$$= a\left(1 - e^2\right)^{1/2}$$



$q = 0$ $\quad r = \frac{\ell}{1+e} = a(1-e)$ periastron

$q = 90^\circ$ $\quad r = \ell = a(1-e^2)$

$q = 180^\circ$ $\quad r = \frac{\ell}{1-e} = a(1+e)$ apastron

Orbital Speed

$$r = \frac{\ell}{1 + e\cos q} \qquad \ell \equiv \frac{L^2}{G\,M} = a\left(1 - e^2\right)$$

$$e = -\frac{G\,M}{2\,\ell}\left(1 - e^2\right) = -\frac{G\,M}{2\,a}$$

Energy:

$$V^2 = 2\,e + \frac{2\,G\,M}{r} = G\,M\left[\frac{2}{r} - \frac{1}{a}\right]$$

Motion in Time

$$r = \frac{\ell}{1 + e\cos q} \qquad \ell \equiv \frac{L^2}{G\,M}$$

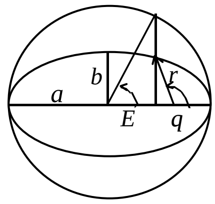
$$\dot{è} = \frac{L}{r^2} = \frac{L}{\ell^2}\,(1 + e\cos q\,)^2$$

$$\frac{dq}{(1 + e\cos q\,)^2} = \frac{L}{\ell^2}\,dt$$

$$\int \frac{dq}{(1 + e\cos q\,)^2} = \frac{L}{\ell^2}\,(t - T)$$

No analytic solution for $q(t)$

Eccentric Anomaly



$b = a\sqrt{1 - e^2} =$

q = true anomaly

E = eccentric anomaly

$x = a\cos E = a\,e + r\cos q$

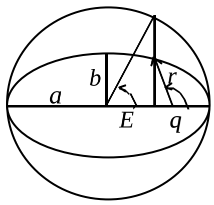
$y = b\sin E = r\sin q$

$r^2 = a^2\,(\cos E - e)^2 + a^2\,(1 - e^2)\sin^2 E$

$r = a\,(1 - e\cos E\,) \rightarrow dr = a\,e\sin E\,dE$

......Kepler's equation giving $E(t)$

Motion in Time



h = mean anomaly

f = orbital phase

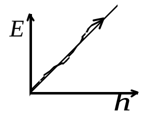
P = orbital period

T = time of periastron passage

Kepler's equation:

$$E - e\sin E = h = 2p\,f = \frac{2p}{P}(t - T)$$

iterate to find $E(t)$



$$\tan\left(\frac{q}{2}\right) = \left(\frac{1+e}{1-e}\right)^{1/2} \tan\left(\frac{E}{2}\right)$$