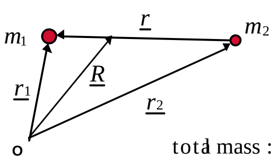
The Two-Body Problem

- Newtonian gravity
- 2 point masses
  - good approx for stars



$$\underline{r} = \underline{r}_1 - \underline{r}_2$$

$$\hat{\underline{r}} = \underline{r} / r$$

total mass : $M = m_1 + m_2$

- O: origin

centre of mass : $\underline{R} = \dfrac{m_1\underline{r}_1 + m_2\underline{r}_2}{M}$

Two-Body motion

- Newtons laws of motion + law of gravitation

Equations of motion:

$$\underline{F}_1 = m_1\ddot{\underline{r}}_1 = \frac{-G m_1 m_2}{r^2}\hat{\underline{r}} \qquad \underline{F}_2 = m_2\ddot{\underline{r}}_2 = -\underline{F}_1$$

add the equations $\quad m_1\ddot{\underline{r}}_1 + m_2\ddot{\underline{r}}_2 = 0$

integrate $dt \quad m_1\dot{\underline{r}}_1 + m_2\dot{\underline{r}}_2 = \underline{A}$

again $\quad m_1\underline{r}_1 + m_2\underline{r}_2 = \underline{A}\,t + \underline{B}$

definition of $\underline{R} \quad M\,\underline{R} = \underline{A}\,t + \underline{B}$

- ∴ Centre of mass moves with constant velocity
  - (unless acted on by an external force)

Relative Motion

- Important for eclipses
- Subtract two equations of motion

$$\ddot{\underline{r}} = \ddot{\underline{r}}_1 - \ddot{\underline{r}}_2 = \frac{-GM}{r^2}\hat{\underline{r}}$$

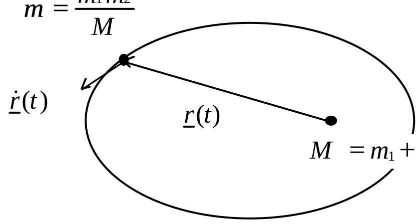
- Multiply by ■ = $m_1 m_2 / M$ = "reduced mass"

$$m\,\ddot{\underline{r}} = \frac{-G\,M\,m}{r^2}\hat{\underline{r}} = \frac{-G\,m_1\,m_2}{r^2}\hat{\underline{r}}$$

- Relative orbit is as if:
  - orbiter has reduced mass ■ = reduced mass
  - stationary central mass is M = total mass.

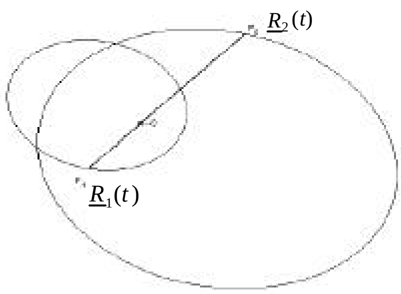
The Relative Orbit

$$m = \frac{m_1 m_2}{M}$$



Important for Eclipses

2 Barycentric Orbits



Important for Radial Velocities

Barycentric Orbits

- Important for radial velocity curves

$m_1\underline{R}_1 + m_2\underline{R}_2 = 0 \quad$ centre of mass frame

$$\underline{r} = \underline{R}_1 - \underline{R}_2 = \frac{m_1 + m_2}{m_2}\underline{R}_1 = \frac{M}{m_2}\underline{R}_1 = -\frac{M}{m_1}\underline{R}_2$$

equations of motion:

$$\ddot{R}_1 = -\frac{G\,m_2}{r^3}\underline{r} \qquad \ddot{R}_2 = -\frac{G\,m_1}{r^3}(-\underline{r})$$

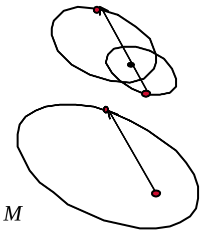
- Eliminate r^3 using $r = f(R_1)$ and $r = f(R_2)$:

$$\ddot{R}_1 = -\frac{G\,m_2^{\,3}}{M^2}\frac{\underline{R}_1}{R_1^{\,3}} \qquad \ddot{R}_2 = -\frac{G\,m_1^{\,3}}{M^2}\frac{\underline{R}_2}{R_2^{\,3}}$$

- the acceleration of each star relative to the centre of mass

Relative vs Barycentric orbits

- (a, e, P, v) relative orbit
- $(a, e, P, v)_{1,2}$ barycentric orbits
- m_1 and m_2 on straight line thru C



$P_1 = P_2 = P \quad e_1 = e_2 = e$

$a_1 = a\, m_2 / M \quad a_2 = a\, m_1 / M$

$a = a_1 + a_2 \quad M = m_1 + m_2$

$a_1 : a_2 : a = V_1 : V_2 : V = m_2 : m_1 : M$

Kepler:

$$\frac{4p^2}{GP^2} = \frac{M}{a^3} = \frac{m_2^3 / M^2}{a_1^3} = \frac{m_1^3 / M^2}{a_2^3}$$

Orbital Speed

orbital speed : $V^2 = \dot{r}^2 + r^2 \dot{q}^2$

conic section : $r = \dfrac{\ell}{1 + e \cos q}$

$$\frac{d}{dt}\left[1 + e\cos q = \frac{\ell}{r}\right] \Rightarrow -e(\sin q)\dot{q} = -\frac{\ell}{r^2}\dot{r}$$

specific angular momentum : $r^2 \dot{q} = L$

$$\therefore \quad \dot{r} = \frac{r^2 \dot{q}}{\ell} e \sin q = \frac{L}{\ell} e \sin q$$

$$r\dot{q} = \frac{L}{r} = \frac{L}{\ell}(1 + e\cos q)$$

Orbital Speed

$$V^2 = \left(\frac{L}{\ell}\right)^2 \left[e^2 \sin^2 q + (1 + e\cos q)^2\right]$$

$$= \left(\frac{L}{\ell}\right)^2 \left[e^2 + 1 + 2e\cos J\right]$$

$$= \left(\frac{L}{\ell}\right)^2 \left[2(e\cos q + 1) + e^2 - 1\right]$$

$$V^2 = \frac{L^2}{\ell}\left[\frac{2}{r} - \frac{1 - e^2}{\ell}\right]$$

$$\ell = \frac{L^2}{G\,M}$$

$$\frac{L}{\ell} = \frac{G\,M}{L}$$

Orbital Speed

$$V^2 = \frac{L^2}{\ell}\left[\frac{2}{r} - \frac{1 - e^2}{\ell}\right]$$

ellipse : $\ell = a(1 - e^2) \quad L^2 = GM\ell$

$$V^2 = GM\left[\frac{2}{r} - \frac{1}{a}\right]$$

ellipse	$e < 1$	$\ell = a(1 - e^2)$	$V^2 = GM\left[\frac{2}{r} - \frac{1}{a}\right]$
circle	$e = 0$	$\ell = a$	$V^2 = \frac{GM}{a}$
parabola	$e = 1$		$V^2 = \frac{2GM}{r}$
hyperbola	$e > 1$	$\ell = a(1 - e^2)$	$V^2 = GM\left[\frac{2}{r} + \frac{1}{a}\right]$

Energy of Orbit

Kinetic energy:

$$KE = T = \frac{1}{2} m_1 V_1^2 + \frac{1}{2} m_2 V_2^2 = \frac{m_1\, m_2}{2\,M} V^2$$

orbital speed: $V^2 = G\,M\left(\dfrac{2}{r} - \dfrac{1}{a}\right)$

Potential energy:

$$PE = W = -\int_r^\infty \frac{G\, m_1\, m_2}{r^2} dr = -\frac{G\, m_1\, m_2}{r}$$

Total energy: $E = T + W = -\dfrac{G\, m_1\, m_2}{2\,a} < 0$

Binding energy: $-E > 0$

Angular momentum of the orbit

- **Angular momentum vector J, defines orbital plane**
  - $\underline{J} = m_1 \underline{L}_1 + m_2 \underline{L}_2$ and $L^2 = G\,M\,a\,(1-e^2)$
    and $L_1^2 = G\,(m_2^3 / M^2)\, a_1\, (1-e^2)$
    and $a_1/a = m_2 / M$
    - same for $\underline{L}_2$
  - hence

$$L_1 = \frac{m_2^2}{M^2} L; \quad L_2 = \frac{m_1^2}{M^2} L$$

therefore

$$J^2 = \frac{G\, m_1^2 m_2^2}{M} a\,(1 - e^2)$$

and the final expression for J is

$$J = \frac{2p\, a^2\, m_1\, m_2}{P\,M}\sqrt{1 - e^2}$$

Orbital Angular momentum

- **Given masses *m1,m2* and Energy *E*,**
  - the angular momentum *J* determines the shape of the orbit
  - ie the eccentricity (or the conic section parameter *l*)

- **For given *E*,**
  - circular orbits have maximum *J*
  - *J* decreases as $e \longrightarrow 1$
  - orbit becomes rectilinear ellipse

- **relation between *E*, and *J* very important in**

  determining when systems interact

  mass exchange and orbital evolution