The Two-Body Problem

- **Newtonian gravity**
- **2 point masses**
  - good approx for stars

\[ r = r_1 - r_2 \]
\[ \hat{r} = \frac{r}{r} \]

**total mass**: \[ M = m_1 + m_2 \]

**centre of mass**: \[ R = \frac{m_1 r_1 + m_2 r_2}{M} \]
Two-Body motion

• **Newton's laws of motion + law of gravitation**

Equations of motion:

\[ F_1 = m_1 \ddot{r}_1 = -\frac{G m_1 m_2}{r^2} \hat{r} \quad F_2 = m_2 \ddot{r}_2 = -F_1 \]

add the equations

\[ m_1 \ddot{r}_1 + m_2 \ddot{r}_2 = 0 \]

integrate \( dt \)

\[ m_1 \dot{r}_1 + m_2 \dot{r}_2 = A \]

again

\[ m_1 r_1 + m_2 r_2 = A t + B \]

definition of \( R \)

\[ M \ R = A t + B \]

\[ M \ R = A t + B \]

• ∴ Centre of mass moves with constant velocity
  - (unless acted on by an external force)
Relative Motion

- Important for eclipses
- Subtract two equations of motion

\[
\ddot{r} = \ddot{r}_1 - \ddot{r}_2 = \frac{-G M}{r^2} \hat{r}
\]

- Multiply by \( \mu = \frac{m_1 m_2}{M} = \text{“reduced mass”} \)

\[
\mu \ddot{r} = -\frac{G M \mu}{r^2} \hat{r} = -\frac{G m_1 m_2}{r^2} \hat{r}
\]

- Relative orbit is as if:
  - orbiter has reduced mass \( \mu = \text{reduced mass} \)
  - stationary central mass is \( M = \text{total mass} \).
The Relative Orbit

\[ \mu = \frac{m_1 m_2}{M} \]

Important for Eclipses

\[ M = m_1 + m_2 \]
2 Barycentric Orbits

Important for Radial Velocities
Barycentric Orbits

• **Important for radial velocity curves**

\[ m_1 R_1 + m_2 R_2 = 0 \quad \text{centre of mass frame} \]

\[ r = R_1 - R_2 = \frac{m_1 + m_2}{m_2} R_1 = \frac{M}{m_2} R_1 = - \frac{M}{m_1} R_2 \]

Equations of motion:

\[ \ddot{R}_1 = -\frac{G m_2}{r^3} r \quad \ddot{R}_2 = -\frac{G m_1}{r^3} (-r) \]

– Eliminate \( r^3 \) using \( r = f(R_i) \) and \( r = f(R_i) \):

\[ \ddot{R}_1 = -\frac{G m_2}{M^2} \frac{R_1}{R_1^3} \quad \ddot{R}_2 = -\frac{G m_1}{M^2} \frac{R_2}{R_2^3} \]

– the acceleration of each star relative to the centre of mass
Relative vs Barycentric orbits

- \( (a, e, P, v) \) relative orbit
- \( (a, e, P, v)_{1,2} \) barycentric orbits
- \( m_1 \) and \( m_2 \) on straight line thru C

\[
P_1 = P_2 = P \quad e_1 = e_2 = e
\]
\[
a_1 = a \frac{m_2}{M} \quad a_2 = a \frac{m_1}{M}
\]
\[
a = a_1 + a_2 \quad M = m_1 + m_2
\]
\[
a_1 : a_2 : a = V_1 : V_2 : V = m_2 : m_1 : M
\]

Kepler:
\[
\frac{4\pi^2}{GP^2} = \frac{M}{a^3} = \frac{m_2^3}{M^2} = \frac{m_1^3}{M^2}
\]
Orbital Speed

orbital speed: \( V^2 = \dot{r}^2 + r^2 \dot{\theta}^2 \)

conic section: \( r = \frac{\ell}{1 + e \cos \theta} \)

\[
\frac{d}{dt} \left[ 1 + e \cos \theta = \frac{\ell}{r} \right] \Rightarrow -e (\sin \theta) \dot{\theta} = -\frac{\ell}{r^2} \dot{r}
\]

specific angular momentum: \( r^2 \dot{\theta} = L \)

\[
\dot{r} = \frac{r^2 \dot{\theta}}{\ell} e \sin \theta = \frac{L}{\ell} e \sin \theta
\]

\[
r \dot{\theta} = \frac{L}{r} = \frac{L}{\ell} (1 + e \cos \theta)
\]
Orbital Speed

\[ V^2 = \left( \frac{L}{\ell} \right)^2 \left[ e^2 \sin^2 \theta + (1 + e \cos \theta)^2 \right] \]

\[ = \left( \frac{L}{\ell} \right)^2 \left[ e^2 + 1 + 2e \cos \theta \right] \]

\[ = \left( \frac{L}{\ell} \right)^2 \left[ 2(e \cos \theta + 1) + e^2 - 1 \right] \]

\[ V^2 = \frac{L^2}{\ell} \left[ \frac{2}{r} - \frac{1 - e^2}{\ell} \right] \]

\[ \ell = \frac{L^2}{G M} \]

\[ \frac{L}{\ell} = \frac{G M}{L} \]
Orbital Speed

\[ V^2 = \frac{L^2}{\ell} \left[ \frac{2}{r} - \frac{1-e^2}{\ell} \right] \]

ellipse: \[ \ell = a(1-e^2) \quad L^2 = GM \ell \]

\[ V^2 = GM \left[ \frac{2}{r} - \frac{1}{a} \right] \]

<table>
<thead>
<tr>
<th>Shape</th>
<th>( e )</th>
<th>( \ell )</th>
<th>( V^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>ellipse</td>
<td>( e &lt; 1 )</td>
<td>( \ell = a(1-e^2) )</td>
<td>( V^2 = GM \left[ \frac{2}{r} - \frac{1}{a} \right] )</td>
</tr>
<tr>
<td>circle</td>
<td>( e = 0 )</td>
<td>( \ell = a )</td>
<td>( V^2 = \frac{GM}{a} )</td>
</tr>
<tr>
<td>parabola</td>
<td>( e = 1 )</td>
<td></td>
<td>( V^2 = \frac{2GM}{r} )</td>
</tr>
<tr>
<td>hyperbola</td>
<td>( e &gt; 1 )</td>
<td>( \ell = a(1-e^2) )</td>
<td>( V^2 = GM \left[ \frac{2}{r} + \frac{1}{a} \right] )</td>
</tr>
</tbody>
</table>
Energy of Orbit

Kinetic energy:

\[ KE = T = \frac{1}{2} m_1 V_1^2 + \frac{1}{2} m_2 V_2^2 = \frac{m_1 m_2}{2 M} V^2 \]

orbital speed:

\[ V^2 = G M \left( \frac{2}{r} - \frac{1}{a} \right) \]

Potential energy:

\[ PE = W = -\int_{r}^{\infty} \frac{G m_1 m_2}{r^2} dr = -\frac{G m_1 m_2}{r} \]

Total energy:

\[ E = T + W = -\frac{G m_1 m_2}{2 a} < 0 \]

Binding energy:

\[ -E > 0 \]
Angular momentum of the orbit

- **Angular momentum vector** $\mathbf{J}$, defines orbital plane
  - $\mathbf{J} = m_1 \mathbf{L}_1 + m_2 \mathbf{L}_2$ and $L^2 = G M a (1-e^2)$
    and $L_1^2 = G \left( \frac{m_2^3}{M^2} \right) a_1 (1-e^2)$
    and $a_1/a = m_2/M$
  - same for $\mathbf{L}_2$
  - hence
    \[
    L_1 = \frac{m_2^2}{M^2} L; \quad L_2 = \frac{m_1^2}{M^2} L
    \]
    therefore
    \[
    J^2 = \frac{G m_1^2 m_2^2}{M} a (1-e^2)
    \]
    and the final expression for $J$ is
    \[
    J = \frac{2\pi}{P} \frac{a^2 m_1 m_2 \sqrt{1-e^2}}{M}
    \]
Orbital Angular momentum

- **Given masses** $m_1, m_2$ and **Energy** $E$,
  - the angular momentum $J$ determines the shape of the orbit
  - i.e. the eccentricity (or the conic section parameter $l$)

- **For given** $E$,
  - circular orbits have maximum $J$
  - $J$ decreases as $e \rightarrow 1$
  - orbit becomes rectilinear ellipse

- **Relation between** $E$, and $J$ very important in determining when systems interact mass exchange and orbital evolution