AS5001 (SUPAAAA) Advanced (Astronomical) Data Analysis
Homework Set 2 (due Mon 24 Oct 2022)

Please provide enough information in your report to make it clear what your algorithms were. Include a copy of your code.

Problem 2.1 Poisson Random Variables

Assume X_i for $i = 1, 2, ...N$ are independent Poisson random variables with mean value $\langle X \rangle = R$. These might be time-binned counts from a light curve, or wavelength-binned counts from the continuum of a spectrum, with R being the mean count rate.

(a) Give an expression for and make a plot of the probability that $X_3 = 1$ as a function of the count rate R. Evaluate $P(\, X_3 > 0 \,|\, R = 0.5\,)$. **[10]**

(b) Derive an expression for the maximum likelihood estimator for R, denoted $R_{\rm ML}$, in terms of the X_i. Similarly, evaluate the expected value and the variance of $R_{\rm ML}$. State whether or not $R_{\rm ML}$ is an unbiased estimator for R, and whether or not it is a minimum variance statistic, and briefly justify your conclusions. **[10]**

Problem 2.2 Continuum and Emission Line

Given $N = 3$ fluxes measured from a spectrum as follows :

$$(F_1, F_2, F_3) = (110 \pm 10, 160 \pm 20, 90 \pm 10) \quad \text{at} \quad (\lambda_1, \lambda_2, \lambda_3) = (80, 105, 120) \ ,$$

fit the function (continuum plus emission line)

$$F(\lambda) = C + A\, G(\lambda), \qquad\qquad G(\lambda) = \frac{1}{\sqrt{2\pi}\Delta} \exp\left\{ -\frac{1}{2} \left(\frac{\lambda - \lambda_0}{\Delta} \right)^2 \right\},$$

where $\lambda_0 = 100$ and $\Delta = 5$ (wavelength of spectral line, resolution of spectrograph) are known (calibrated nuisance parameters), and the parameters of interest in the fit are C (continuum flux density) and A (integrated emission-line flux) .

(a) Derive general expressions for the maximum likelihood estimators $C_{\rm ML}$ and $A_{\rm ML}$, and for their variances, in terms of the data F_i, error bars σ_i and pattern $G_i \equiv G(\lambda_i)$. Evaluate these for the specific dataset above to determine the best-fit parameters and their 1–parameter 1–σ confidence limits. Plot the data points, error bars, and best-fit model as functions of λ. Offset each parameter by its uncertainty, $\pm\sigma$, and plot the 4 corresponding models (on the same plot) for comparison with the data. **[10]**

(b) Make a second plot showing χ^2 contours in the C vs A plane corresponding to the 2–parameter 1–σ, 2–σ, and 3–σ confidence regions (e.g. using PGCONT). Show also the $\Delta\chi^2 = 1$ contour. Conclude from the shape of the contours whether C and A are independent, or positively or negatively correlated. Draw lines on the plot identifying the 1–parameter 1–σ confidence intervals (e.g. using PGMOVE and PGDRAW), and comment on their relation to the $\Delta\chi^2 = 1$ contour. **[10]**

(c) Give a general expression for the Hessian matrix for this fit, and for its inverse. Evaluate the inverse of the Hessian matrix to obtain the parameter variances and covariances. Evaluate the correlation coefficient between $C_{\rm ML}$ and $A_{\rm ML}$. Does this result confirm your conclusion from (b)? **[10]**

(d) Construct an orthogonal basis for this fit, and plot the two basis functions versus λ. Plot χ^2 contours using the orthogonal basis, and compare with your results in (b). **[5]**

Problem 2.3 Mass of Black Hole in V404 Cygni

The X-ray nova V404 Cyg is a binary system suspected to harbour a black hole. The K0 IV companion star and black hole candidate orbit about their common centre of mass with a period $P = 6.4714 \pm 0.0001$ days. Radial velocity measurements of the companion star's absorption lines yield a sinusoidal radial velocity curve with semi-amplitude

$$K_C = 208.5 \pm 0.7 \text{ km s}^{-1}.$$

The companion star fills its Roche lobe and co-rotates with the binary orbit. Its spectral lines are thereby rotationally broadened by

$$(V_{\rm rot} \sin i) = 38.8 \pm 1.1 \text{ km s}^{-1}.$$

The inclination i (angle between the line of sight and the vector normal to the binary orbit plane) is not well determined, but the absence of eclipses and presence of 0.2 mag ellipsoidal variations from the companion star imply

$$50^\circ < i < 80^\circ.$$

Use a Monte-Carlo simulation to investigate posterior probability distributions for the masses M_X of the black hole candidate (X-ray source), and M_C of the companion star. Use the above observational constraints on P, K_C, $(V_{\rm rot} \sin i)$, and i, and the following binary star formulae :

$$\left(\frac{2\pi}{P}\right)^2 = \frac{G\,(M_X + M_C)}{a^3} \qquad\qquad (K_X + K_C) = \frac{2\pi\, a\, \sin i}{P}$$

$$\frac{K_X}{K_C} = \frac{M_C}{M_X} = q \qquad\qquad (V_{\rm rot} \sin i) = 0.462\, K_C\, q^{1/3}\,(1+q)^{2/3}$$

Here a is the separation between the two stars, and $q = M_C/M_X$ is the mass ratio.

(a) Use the above formulae to solve for the masses M_X and M_C in terms of the observed quantities P, K_C, $(V_{\rm rot} \sin i)$, and the inclination i. To do this, first derive the mass function

$$F_X \equiv \frac{M_X\,(\sin i)^3}{(1+q)^2} = \frac{K_C^3\, P}{2\pi\, G}.$$

Why is F_X a lower limit on M_X? Next note that K_C and $(V_{\rm rot} \sin i)$ together determine q, and use this to eliminate q from the mass function, so that you can calculate M_X and M_C from the three observed quantities P, K_C, and $(V_{\rm rot}\ \sin i)$, plus $\sin i$. **[15]**

(b) Use your RANG Gaussian random number subroutine to generate mock datasets that sample the observed distributions of P, K_C and $(V_{\rm rot} \sin i)$. Use a distribution uniform in $\cos i$ to sample the unknown orbit orientation at random between the observational upper and lower limits on i. For each mock dataset, use the expressions you derived in (a) to calculate the mass function, and the corresponding values of M_X and M_C. Express masses in units of the Sun's mass $M_\odot$. Do this for a large number of mock datasets (trials), and make histogram plots (e.g. using PGHIST) of the resulting posterior mass distributions. Average over the trials to estimate mean values and standard deviations for M_X and M_C. Evaluate also the 1–parameter 1–σ confidence intervals, noting that these are asymmetric because the posterior distributions are skewed, and compare these with the standard deviations. Evaluate the probability that $M_X > 3\,M_\odot$, and use the result to answer the question "Is there a black hole in this system"? **[15]**

(c) Plot the joint probability distribution of M_X and M_C. One way to do this is to use your Monte Carlo trials, and just "plot the dots" for each trial value of M_C and M_X. Another way is to accumulate a large number of trial values into a grid spanning some appropriate range in M_C vs M_X, and then plot the array as contours (e.g. using PGCONT) and/or a greyscale image (e.g. using PGGRAY). Select the appropriate contour levels, and outline on your plot the 2–parameter 1–σ and 2–σ confidence regions. Plot also the 1–parameter confidence intervals found in (b). Are M_X and M_C independent? Evaluate the correlation coefficient between M_X and M_C. **[15]**