Summary: Fourier Analysis

$$\text{model}: \quad \mu(t) = \mu_0 + \sum_k c_k \cos\omega_k t + s_k \sin\omega_k t$$

$$\text{even spacing}: \quad t_i = t_0 + i\,\Delta t \qquad T = N\,\Delta t \qquad i = 1,2,...,N$$

$$\text{Fourier frequencies}: \quad \omega_k = k\,\Delta\omega = 2\pi / P_k \qquad P_k = T/k \qquad k = 0,1,...,K_{\max} = N/2$$

$$\text{Nyquist frequency}: \quad \omega_{Nyq} = \pi/\Delta t = 2\pi/P_{Nyq} \qquad P_{Nyq} = 2\,\Delta t$$

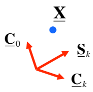
$$\text{Orthogonal basis}: \quad \underline{\mathbf{C}}_k = \cos\omega_k \underline{\mathbf{t}} \qquad \underline{\mathbf{S}}_k = \sin\omega_k \underline{\mathbf{t}}$$

$$\text{Model}: \quad \underline{\mu} = \mu_o \underline{\mathbf{C}}_0 + \sum_{k=1} \left(c_k \underline{\mathbf{C}}_k + s_k \underline{\mathbf{S}}_k \right)$$

Exact fit possible by using N parameters to fit N data points.

$$\text{Badness-of-fit}: \quad \chi^2 = \left\| \underline{\mathbf{X}} - \underline{\mu} \right\|$$

$$\text{Optimal fit}: \quad \hat{\mu}_0 = \frac{\underline{\mathbf{X}} \bullet \underline{\mathbf{C}}_0}{\underline{\mathbf{C}}_0 \bullet \underline{\mathbf{C}}_0} \qquad \hat{c}_k = \frac{\underline{\mathbf{X}} \bullet \underline{\mathbf{C}}_k}{\underline{\mathbf{C}}_k \bullet \underline{\mathbf{C}}_k} \qquad \hat{s}_k = \frac{\underline{\mathbf{X}} \bullet \underline{\mathbf{S}}_k}{\underline{\mathbf{S}}_k \bullet \underline{\mathbf{S}}_k}$$



$$\text{Power spectrum}: \quad P(\omega_k) = \hat{A}_k^2 = \hat{c}_k^2 + \hat{s}_k^2$$

Decomposes lightcurve into frequency components.

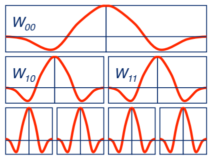
Wavelet Analysis - Wavelet Basis Functions

Fourier basis isolates in frequency but not in time.
Wavelet basis isolates in both frequency and time.

Exact fit possible by using N parameters to fit N data points.

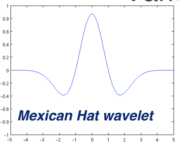
$$W_{k\,j}(x) = W\left[2^k (x - j) \right]$$
$$j = 0, ..., \left(2^k - 1 \right)$$

Many wavelet shapes possible. e.g. "Mexican Hat" wavelet

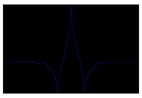


At each new level, k -> k+1:
Double the wavelet frequency.
Double the number of wavelets.
Complete orthogonal basis.

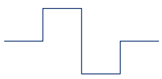
Various Wavelet Shapes



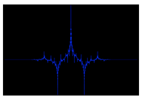
Mexican Hat wavelet

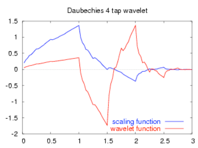


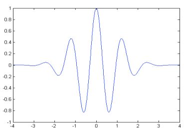
Coiflet



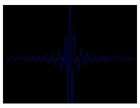
Haar wavelet







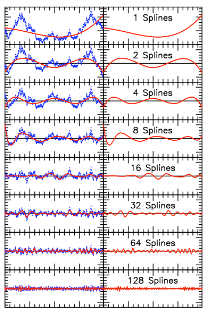
Morlet wavelet



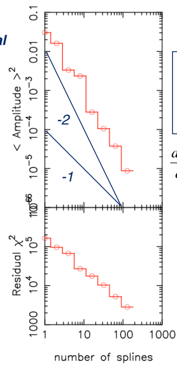
Shannon wavelet

Spline Decomposition - Red Noise

Fit and subtract sequence of cubic splines.



Orthogonal spline basis.



Red noise power spectrum.

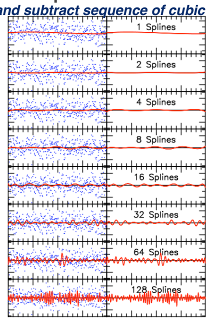
Power law

$$\frac{d\,Power}{d\ln(\omega)} \propto \omega^{-2}$$

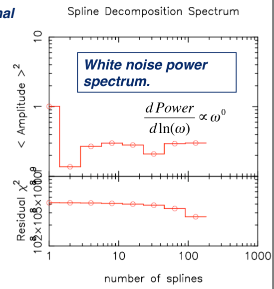
Red Noise: Random walk

Spline Decomposition - White Noise

Fit and subtract sequence of cubic splines.



Orthogonal spline basis.



White noise power spectrum.

$$\frac{d\,Power}{d\ln(\omega)} \propto \omega^{0}$$

White Noise: Independent noise at each time

Cross-correlation

- **Cross-correlation function** (CCF) used to measure the position (and strength) of a feature in the data.
- Pattern *P(x)* matched in width (and shape) to the feature being measured.
- Shift the pattern by an offset *s*, and scale it to fit the data D_i with error bars σ_i measured at positions X_i :

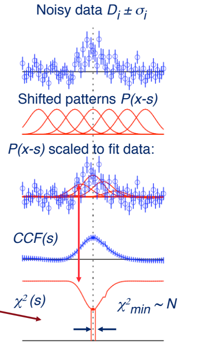
$$CCF(s) = \frac{\sum_i P(X_i - s)\, D_i / \sigma_i^2}{\sum_i P^2(X_i - s) / \sigma_i^2}$$

Optimal scaling, yet again!

$$\mathrm{Var}\left[CCF(s) \right] = \frac{1}{\sum_i P^2(X_i - s) / \sigma_i^2}$$

Note CCF errors are correlated.

- Good fit: $\chi^2_{min} \sim N$. Get *σ(s)* from $\Delta\chi^2 = 1$.
- CCF analysis fits a non-linear model to the data. **Should minimise χ^2, rather than maximising CCF.**



1σ error bar: $\Delta\chi^2 = \chi^2 - \chi^2_{min} = 1$

Pattern too narrow

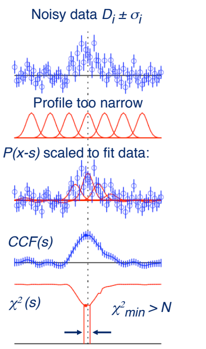
- Gaussian feature and pattern:

$$\mu_i = \langle D_i \rangle = A\exp\left\{-\frac{X_i^2}{2\Delta_0^2}\right\} \qquad P(x) = \exp\left\{-\frac{x^2}{2\Delta^2}\right\}$$

- Pattern width Δ **narrower** than the width Δ_0 of the feature in the data.

$$\langle CCF(s)\rangle \approx \left(\frac{2A^2}{1+(\Delta/\Delta_0)^2}\right)^{1/2}\exp\left\{-\frac{s^2}{2\left(\Delta^2+\Delta_0^2\right)}\right\}$$

- CCF then has **larger error bars**, a **shorter correlation length,** and a **higher** but **narrower peak.**
- Poor fit: χ^2 minimum narrower but shallow.
- **Larger error bar** on *s*.



Wider 1σ error bar: $\Delta\chi^2 = \chi^2 - \chi^2_{min} = 1$

Pattern too wide

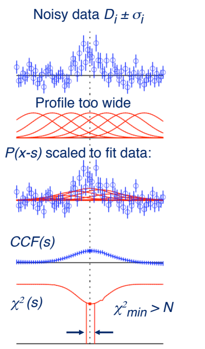
- Gaussian feature and pattern:

$$\mu_i = \langle D_i \rangle = A\exp\left\{-\frac{X_i^2}{2\Delta_0^2}\right\} \qquad P(x) = \exp\left\{-\frac{x^2}{2\Delta^2}\right\}$$

- Pattern width Δ **wider** than the width Δ_0 of the feature in the data.

$$\langle CCF(s)\rangle \approx \left(\frac{2A^2}{1+(\Delta/\Delta_0)^2}\right)^{1/2}\exp\left\{-\frac{s^2}{2\left(\Delta^2+\Delta_0^2\right)}\right\}$$

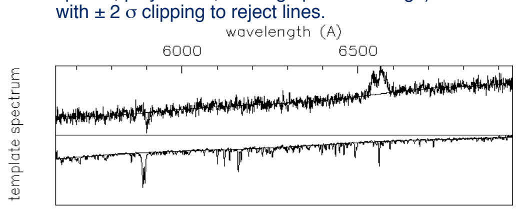
- CCF then has **smaller error bars**, **stronger correlations**, and **lower** but **wider peak**.
- Poor fit: χ^2 minimum wide and shallow.
- **Larger error bar** on *s*.



Wider 1σ error bar: $\Delta\chi^2 = \chi^2 - \chi^2_{min} = 1$

Cross-Correlation Radial Velocities

- Data: spectrum of black-hole binary candidate GRO J0422+32
- Pattern: "template" spectrum of normal K5V star of known radial velocity.
- Mask Hα emission line. Fit continuum (e.g. splines, polynomial, running optimal average) with $\pm$ 2 σ clipping to reject lines.

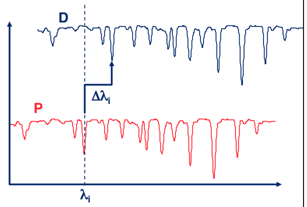


Wavelength and velocity shifts

- Target spectrum D_i is measured at wavelengths λ_i and has associated errors σ_i.
- Template spectrum P is measured on same (or very similar) wavelength grid. Errors assumed negligible.
- For small velocity shift Δv: $\Delta\lambda_i = \frac{\lambda_i \Delta v}{c}$

Interpolate

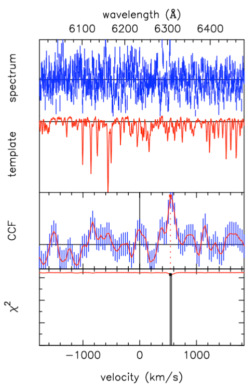
$$CCF(\Delta v) = \frac{\sum_i P(\lambda_i - \Delta\lambda_i) D_i / \sigma_i^2}{\sum_i \left[P(\lambda_i - \Delta\lambda_i)\right]^2 / \sigma_i^2}$$



Note that since *D* is redshifted relative to *P* in this example, *CCF (Δv)* will produce a peak at positive *Δv*.

Radial velocity of GRO J0422+32

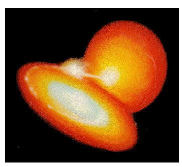
- Subtract continuum fit.
- Cross-correlate data with template spectrum.
- Compute CCF for shifts in range $\pm$ 1800 km s^{-1}.
- CCF shows peak between 500 and 600 km s^{-1}.
- Use $\Delta\chi^2 = 1$ for 1σ error bar on radial velocity.

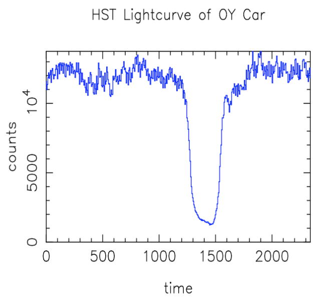


Project 1 = HST lightcurve of OY Car

2 oscillations present.

Amplitude and phase modulated by eclipse.





Project 2 = Keck Spectra of a Black-Hole Binary

13 spectra from Keck 10m on Mauna Kea.

Fit continuum.

Cross-correlate with template star spectra.

Measure 13 radial velocities.

Fit sine curve to measure velocity semi-amplitude.

Work out constraints on the black hole mass.