Summary: Fourier Analysis

$$\text{model}: \quad \mu(t) = \mu_0 + \sum_k c_k \cos\omega_k t + s_k \sin\omega_k t$$

$$\text{even spacing}: \quad t_i = t_0 + i\,\Delta t \qquad T = N\,\Delta t \qquad i = 1,2,\ldots,N$$

$$\text{Fourier frequencies}: \quad \omega_k = k\,\Delta\omega = 2\pi / P_k \quad P_k = T/k \quad k = 0,1,\ldots,K_{\max} = N/2$$

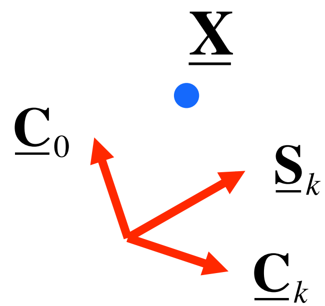
$$\text{Nyquist frequency}: \quad \omega_{Nyq} = \pi/\Delta t = 2\pi / P_{Nyq} \quad P_{Nyq} = 2\,\Delta t$$

$$\text{Orthogonal basis}: \quad \underline{\mathbf{C}}_k = \cos\omega_k \underline{\mathbf{t}} \quad \underline{\mathbf{S}}_k = \sin\omega_k \underline{\mathbf{t}}$$

$$\text{Model}: \quad \underline{\mu} = \mu_o \underline{\mathbf{C}}_0 + \sum_{k=1} \left(c_k \underline{\mathbf{C}}_k + s_k \underline{\mathbf{S}}_k \right)$$

Exact fit possible by using N parameters to fit N data points.

$$\text{Badness-of-fit}: \quad \chi^2 = \left\| \underline{\mathbf{X}} - \underline{\mu} \right\|$$



$$\text{Optimal fit}: \quad \hat{\mu}_0 = \frac{\underline{\mathbf{X}} \bullet \underline{\mathbf{C}}_0}{\underline{\mathbf{C}}_0 \bullet \underline{\mathbf{C}}_0} \quad \hat{c}_k = \frac{\underline{\mathbf{X}} \bullet \underline{\mathbf{C}}_k}{\underline{\mathbf{C}}_k \bullet \underline{\mathbf{C}}_k} \quad \hat{s}_k = \frac{\underline{\mathbf{X}} \bullet \underline{\mathbf{S}}_k}{\underline{\mathbf{S}}_k \bullet \underline{\mathbf{S}}_k}$$

$$\text{Power spectrum}: \quad P(\omega_k) = \hat{A}_k^{\,2} \equiv \hat{c}_k^{\,2} + \hat{s}_k^{\,2}$$

Decomposes lightcurve into frequency components.

Wavelet Analysis - Wavelet Basis Functions

Fourier basis isolates in frequency but not in time.

Wavelet basis isolates in both frequency and time.

Exact fit possible by using N parameters to fit N data points.

$$W_{k\,j}(x) = W\left[2^k(x-j)\right]$$

$$j = 0, \ldots, \left(2^k - 1\right)$$

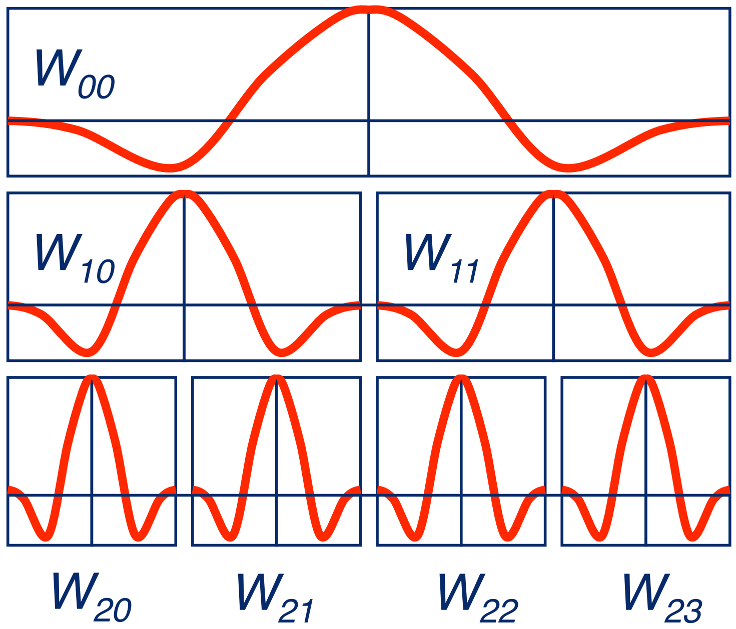
At each new level, k -> k+1:

Double the wavelet frequency.

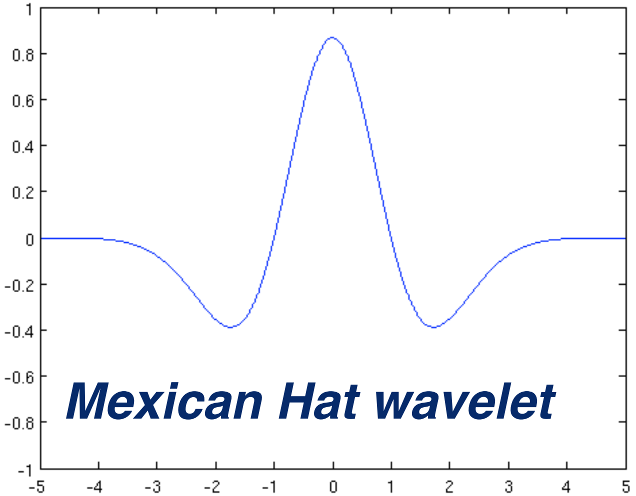
Double the number of wavelets.

Complete orthogonal basis.

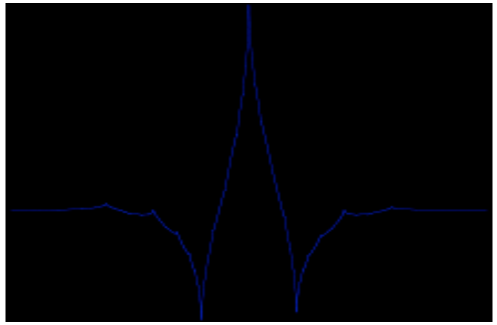
Many wavelet shapes possible. e.g. "Mexican Hat" wavelet



Various Wavelet Shapes

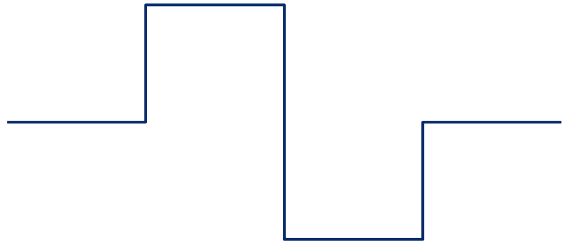


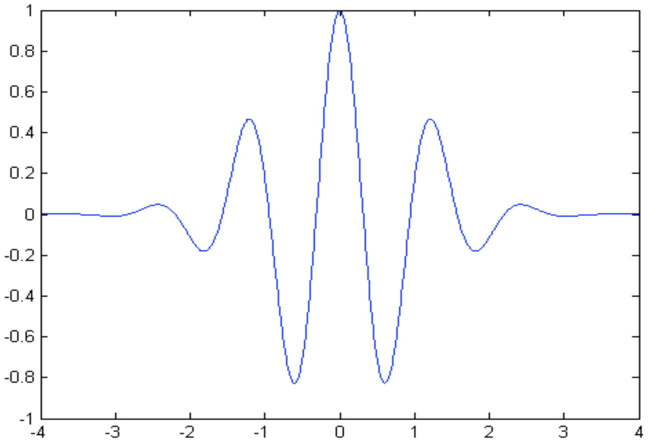
Mexican Hat wavelet



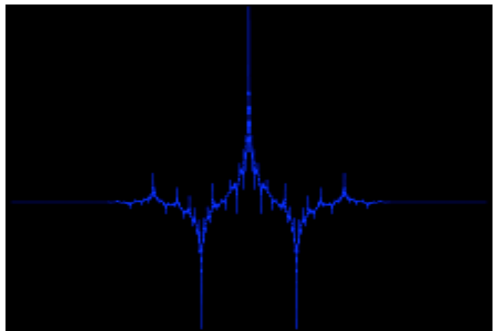
Coiflet

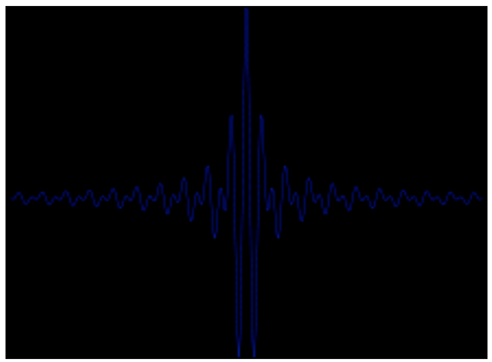
Haar wavelet



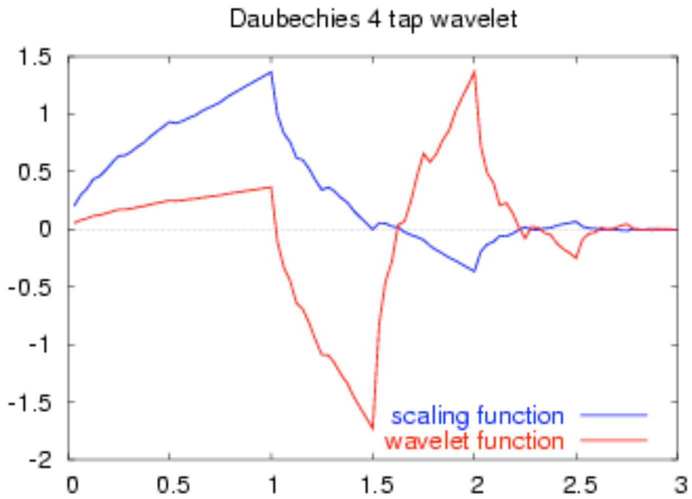


Morlet wavelet



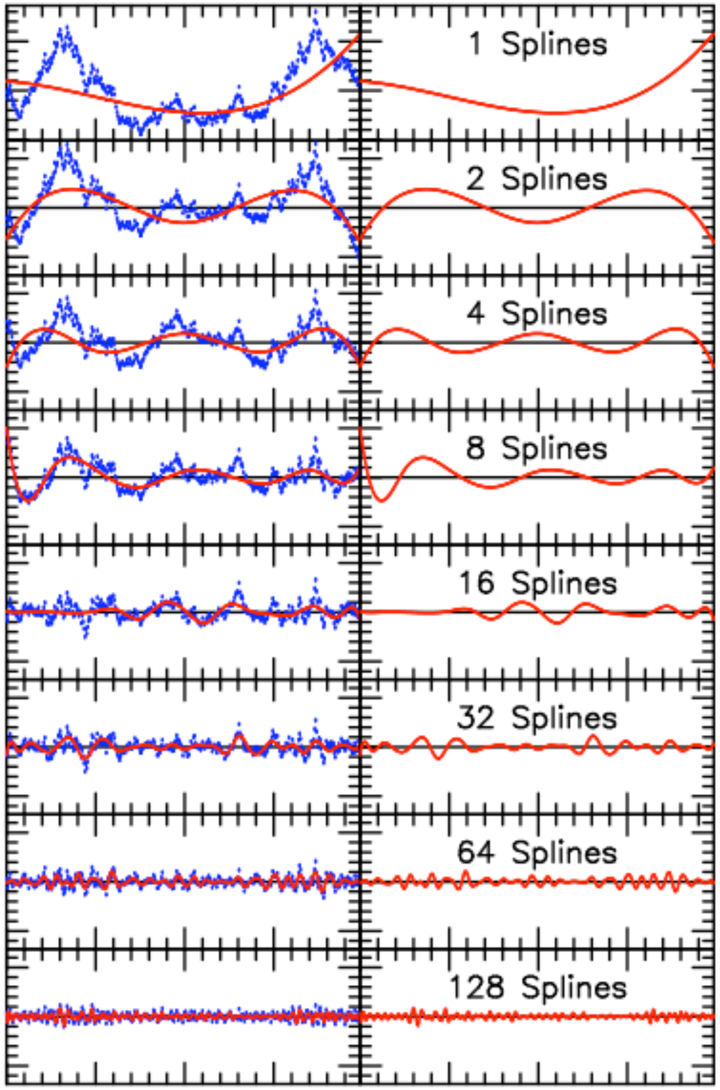


Shannon wavelet



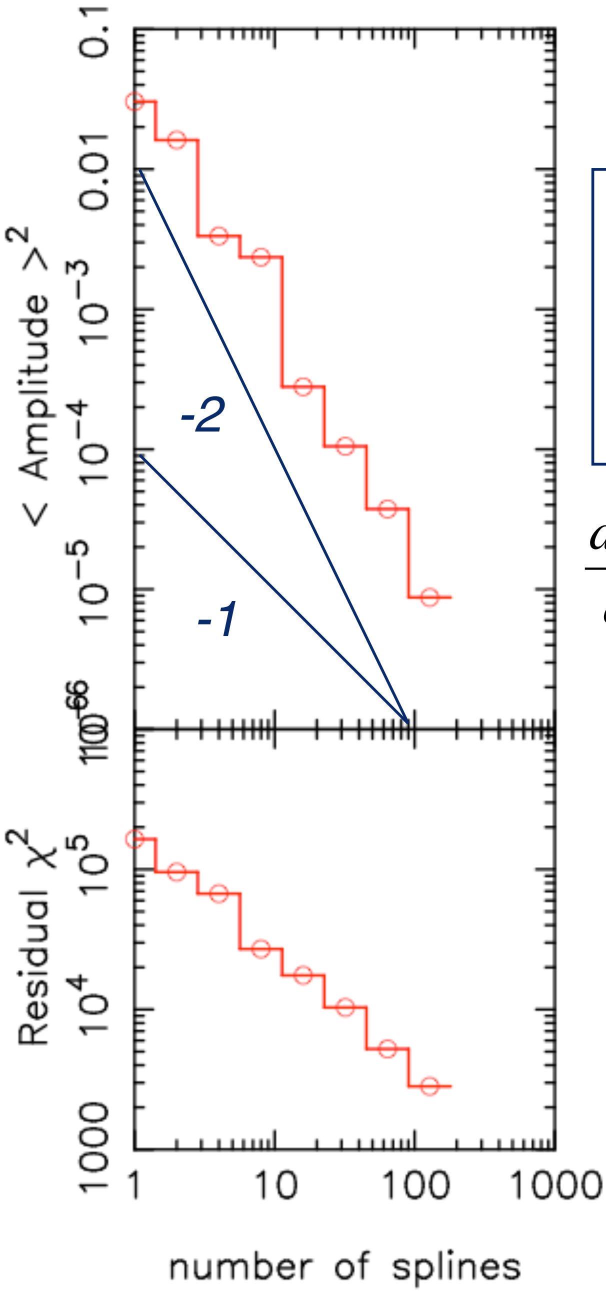
Spline Decomposition - Red Noise

Fit and subtract sequence of cubic splines.



Orthogonal spline basis.

Red Noise: Random walk



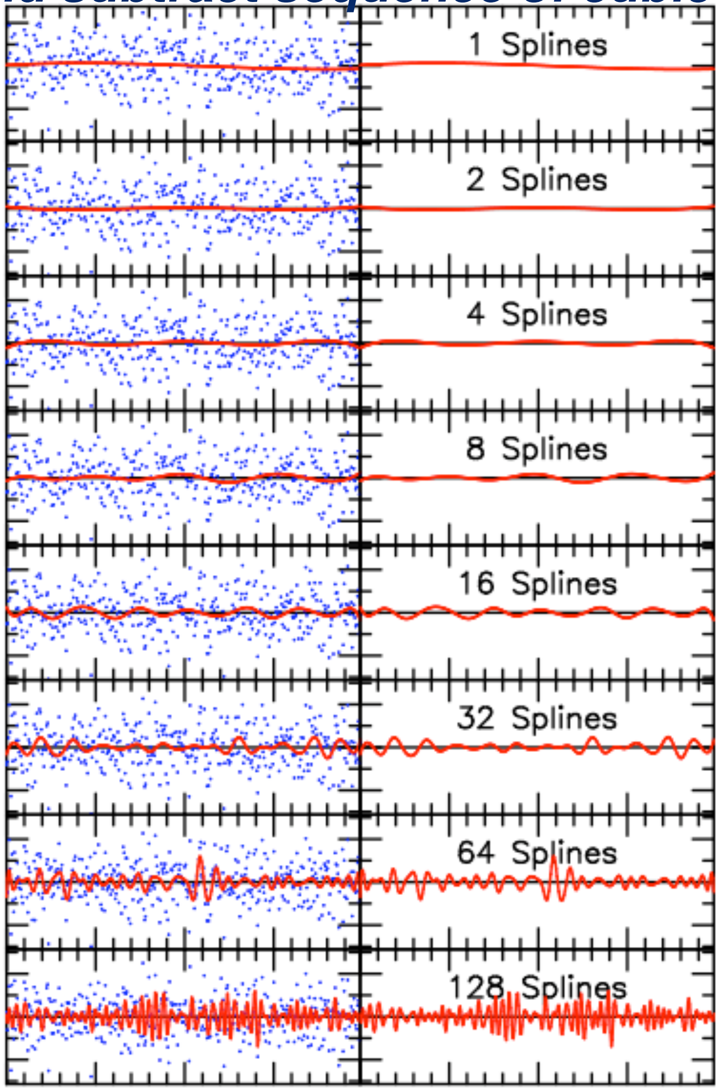
Red noise power spectrum.

Power law

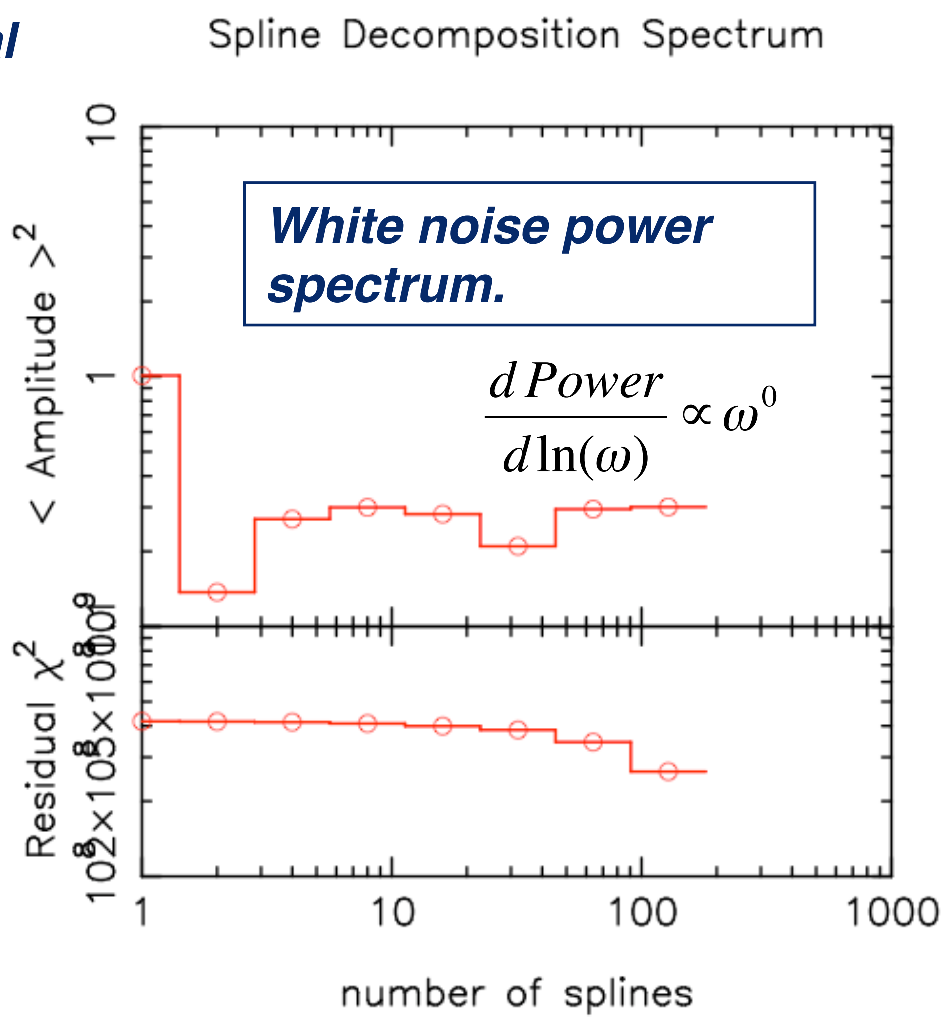
$$\frac{d\,Power}{d\ln(\omega)} \propto \omega^{-2}$$

Spline Decomposition - White Noise

Fit and subtract sequence of cubic splines.



Orthogonal spline basis.



White noise power spectrum.

$$\frac{d\,Power}{d\ln(\omega)} \propto \omega^0$$

White Noise: Independent noise at each time

Cross-correlation

- **Cross-correlation function** (CCF) used to measure the position (and strength) of a feature in the data.
- Pattern *P(x)* matched in width (and shape) to the feature being measured.
- Shift the pattern by an offset *s*, and scale it to fit the data D_i with error bars σ_i measured at positions X_i :

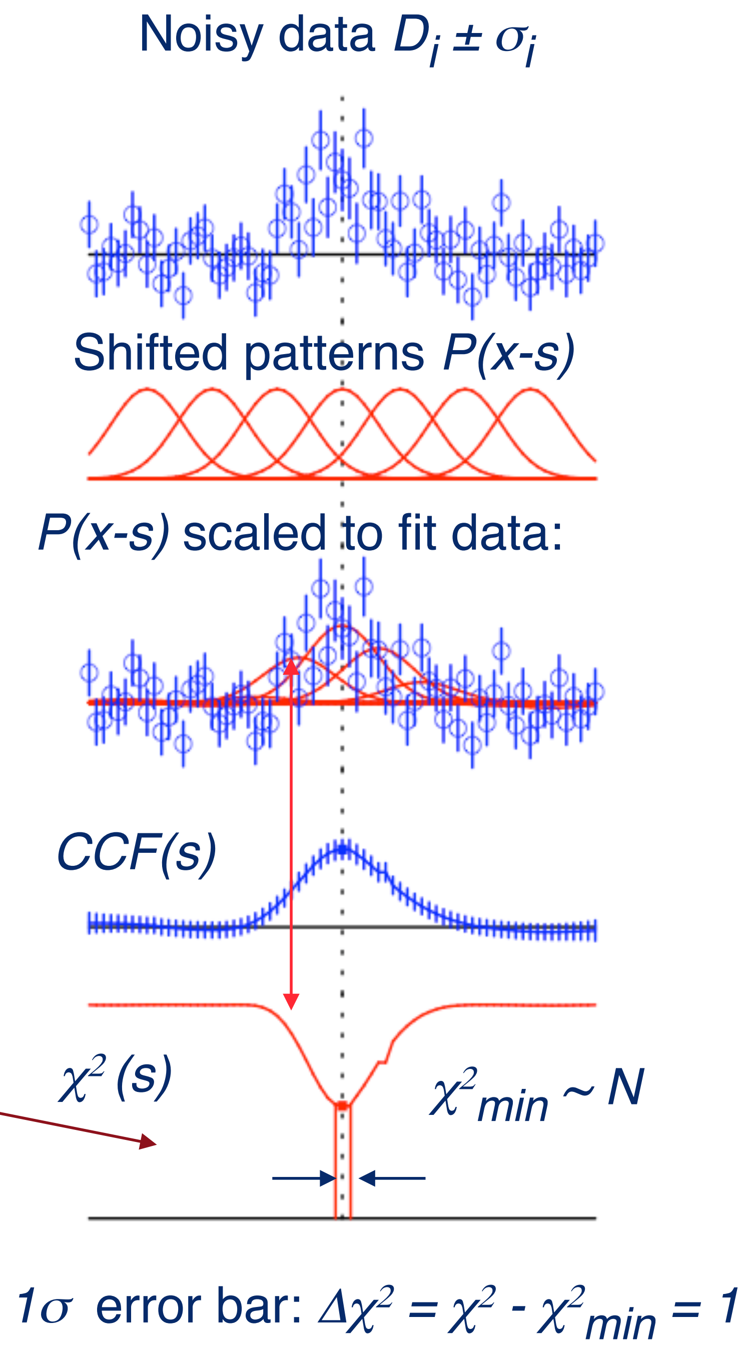
$$CCF(s) \equiv \frac{\sum_i P(X_i - s) D_i / \sigma_i^2}{\sum_i P^2(X_i - s)/\sigma_i^2}$$

Optimal scaling, yet again!

$$\mathrm{Var}\left[CCF(s)\right] = \frac{1}{\sum_i P^2(X_i - s)/\sigma_i^2}$$

Note CCF errors are correlated.

- Good fit: $\chi^2_{min} \sim N$. Get $\sigma(s)$ from $\Delta\chi^2 = 1$.
- CCF analysis fits a non-linear model to the data**. Should minimise χ^2, rather than maximising CCF.**



Pattern too narrow

- Gaussian feature and pattern:

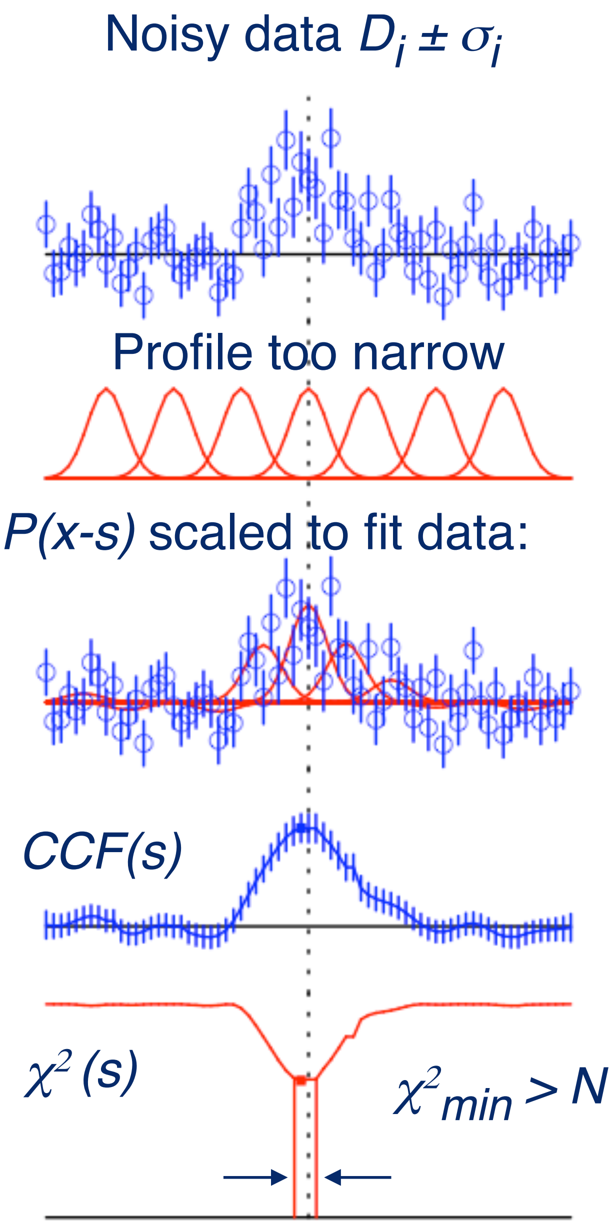
$$\mu_i = \langle D_i \rangle = A \exp\left\{ -\frac{X_i^2}{2\Delta_0^2} \right\} \qquad P(x) = \exp\left\{ -\frac{x^2}{2\Delta^2} \right\}$$

- Pattern width Δ **narrower** than the width Δ_0 of the feature in the data.

$$\langle CCF(s) \rangle \approx \left(\frac{2A^2}{1 + (\Delta/\Delta_0)^2} \right)^{1/2} \exp\left\{ -\frac{s^2}{2\left(\Delta^2 + \Delta_0^2\right)} \right\}$$

- CCF then has **larger error bars**, a **shorter correlation length,** and a **higher** but **narrower peak.**

- Poor fit: χ^2 minimum narrower but shallow.
- **Larger error bar** on *s*.



Wider 1σ error bar: $\Delta\chi^2 = \chi^2 - \chi^2_{min} = 1$

Pattern too wide

- Gaussian feature and pattern:

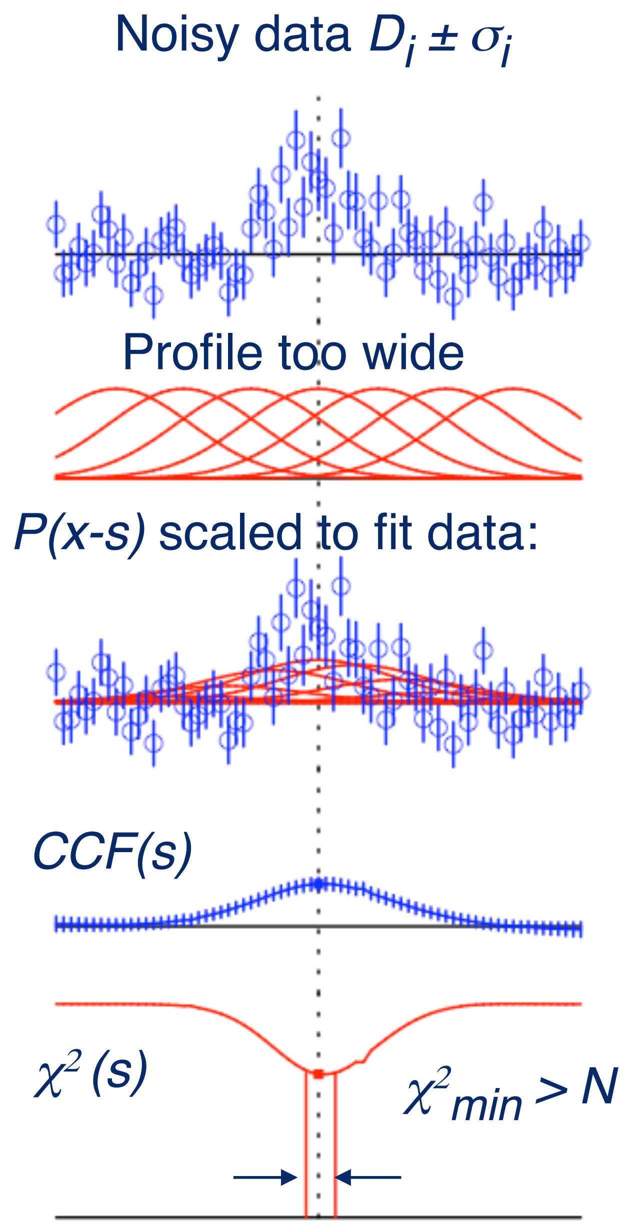
$$\mu_i = \langle D_i \rangle = A \exp\left\{ -\frac{X_i^2}{2\Delta_0^2} \right\} \qquad P(x) = \exp\left\{ -\frac{x^2}{2\Delta^2} \right\}$$

- Pattern width Δ **wider** than the width Δ_0 of the feature in the data.

$$\langle CCF(s) \rangle \approx \left(\frac{2A^2}{1 + (\Delta/\Delta_0)^2} \right)^{1/2} \exp\left\{ -\frac{s^2}{2\left(\Delta^2 + \Delta_0^2\right)} \right\}$$

- CCF then has **smaller error bars**, **stronger correlations**, and **lower** but **wider peak**.

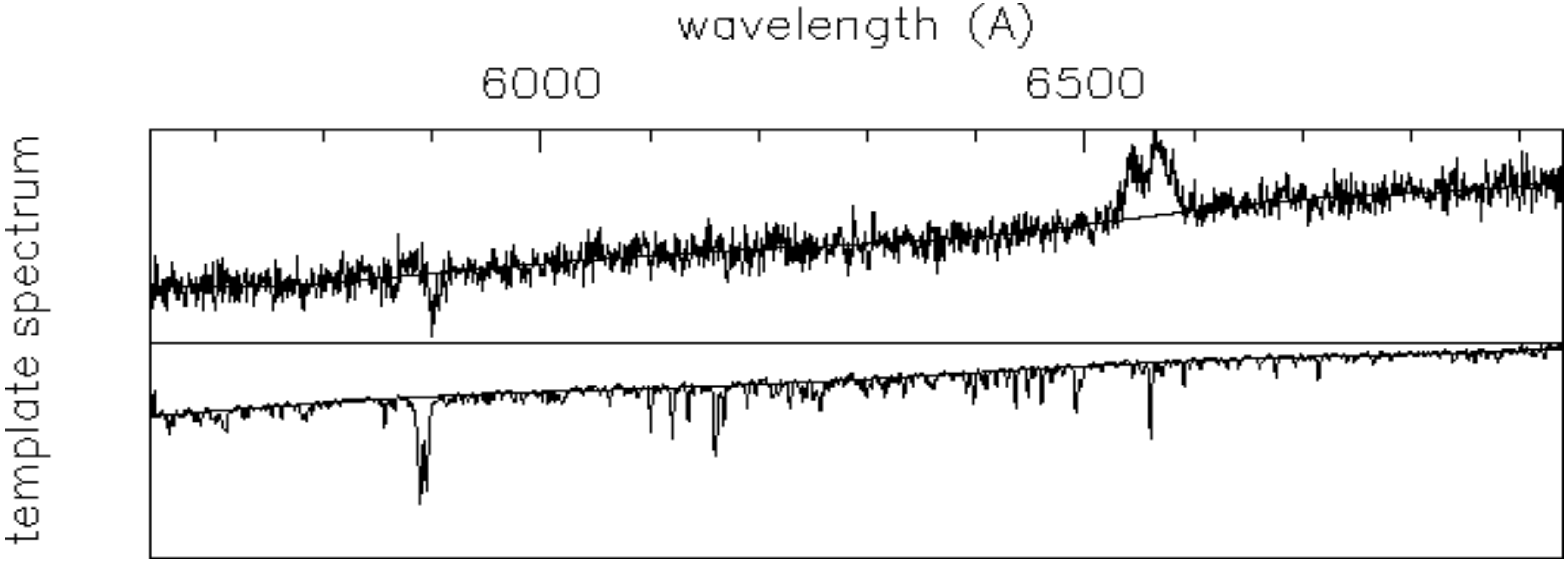
- Poor fit: χ^2 minimum wide and shallow.
- **Larger error bar** on *s*.



Wider 1σ error bar: $\Delta\chi^2 = \chi^2 - \chi^2_{min} = 1$

Cross-Correlation Radial Velocities

- Data: spectrum of black-hole binary candidate GRO J0422+32
- Pattern: "template" spectrum of normal K5V star of known radial velocity.
- Mask Hα emission line. Fit continuum (e.g. splines, polynomial, running optimal average) with ± 2 σ clipping to reject lines.

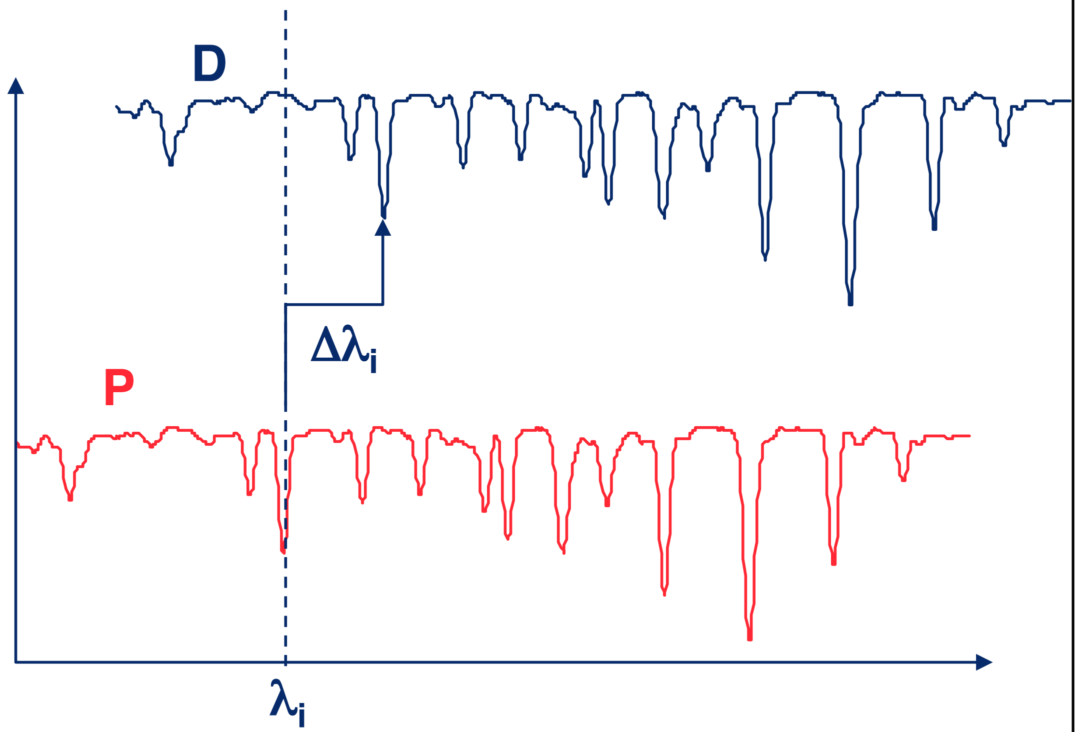


Wavelength and velocity shifts

- Target spectrum D_i is measured at wavelengths λ_i and has associated errors σ_i.
- Template spectrum P is measured on same (or very similar) wavelength grid. Errors assumed negligible.
- For small velocity shift Δv: $$\Delta\lambda_i = \frac{\lambda_i \Delta v}{c}$$

Interpolate

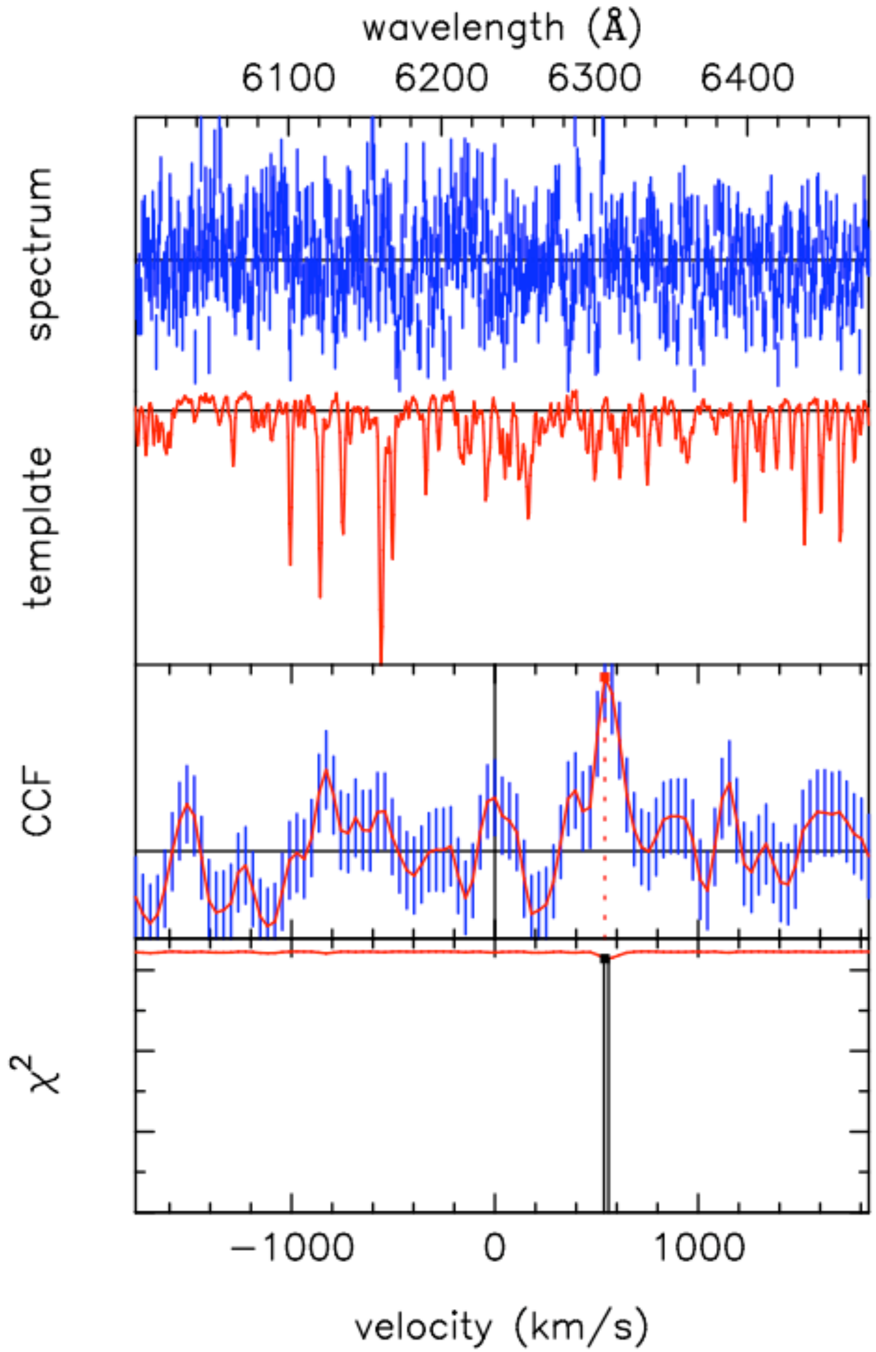
$$CCF(\Delta v) = \frac{\sum_i P(\lambda_i - \Delta\lambda_i) D_i / \sigma_i^2}{\sum_i \left[P(\lambda_i - \Delta\lambda_i)\right]^2 / \sigma_i^2}$$



Note that since *D* is redshifted relative to *P* in this example, *CCF (Δv)* will produce a peak at positive *Δv*.

Radial velocity of GRO J0422+32

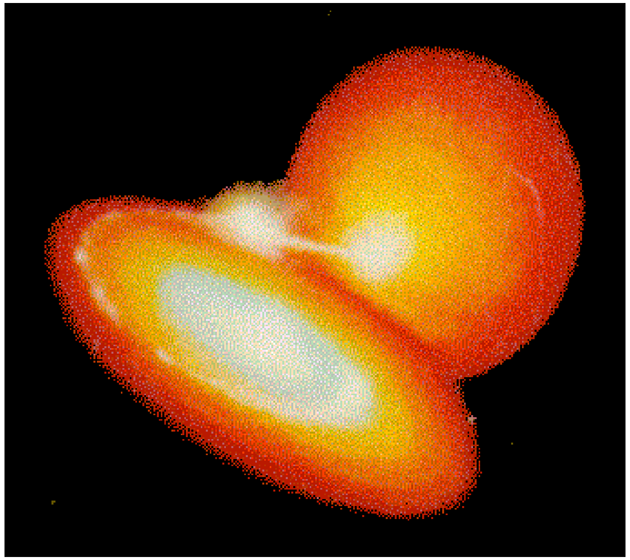
- Subtract continuum fit.
- Cross-correlate data with template spectrum.
- Compute CCF for shifts in range ± 1800 km s^{-1}.
- CCF shows peak between 500 and 600 km s^{-1}.
- Use $\Delta\chi^2 = 1$ for 1σ error bar on radial velocity.

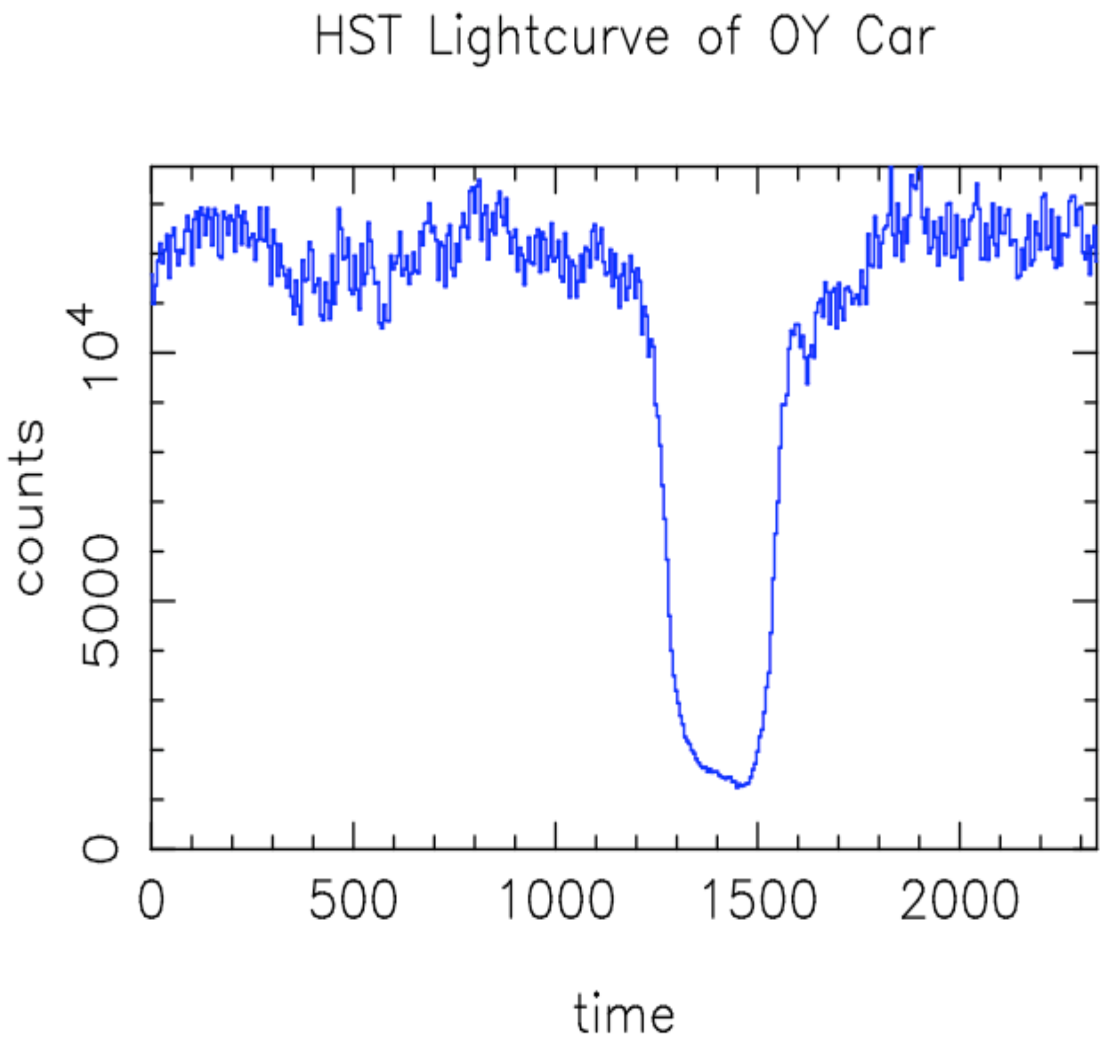


Project 1 = HST lightcurve of OY Car

2 oscillations present.

Amplitude and phase modulated by eclipse.





Project 2 = Keck Spectra of a Black-Hole Binary

13 spectra from Keck 10m on Mauna Kea.

Fit continuum.

Cross-correlate with template star spectra.

Measure 13 radial velocities.

Fit sine curve to measure velocity semi-amplitude.

Work out constraints on the black hole mass.