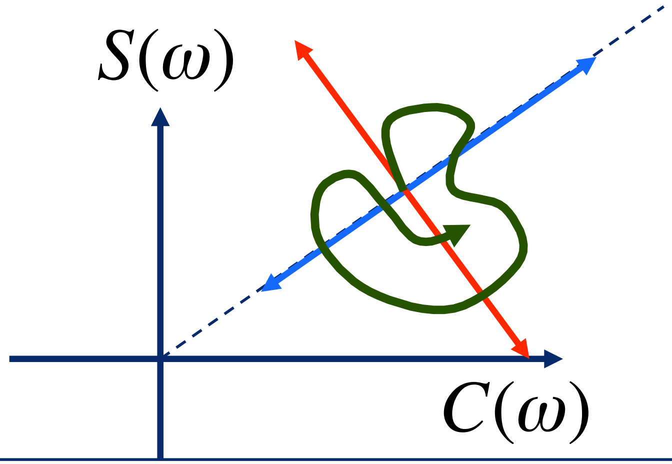
Unstable Oscillations

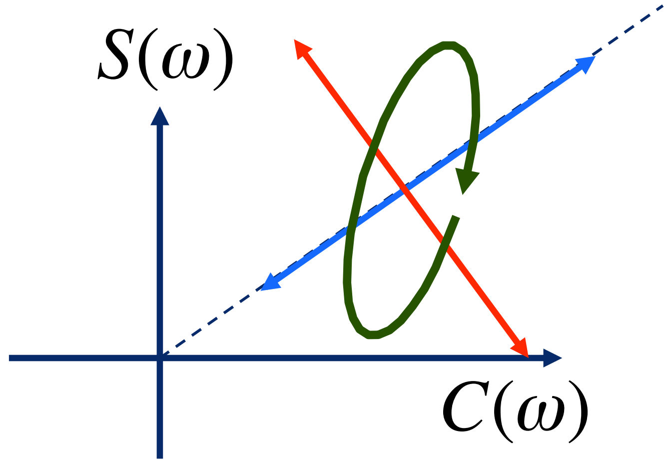
$$X(t) = [A_0 + A(t)] \sin(\omega t + \Phi_0 + \Phi(t))$$

***Amplitude modulation**: $A(t)$*

***Phase modulation**: $\Phi(t)$*



Special case:



Unstable oscillation
has a **broader periodogram peak.**

Amplitude drifts -> sidelobes

Phase drifts equivalent to frequency ω changing with time.

$$A(t) = A \sin \Omega t + B \cos \Omega t$$

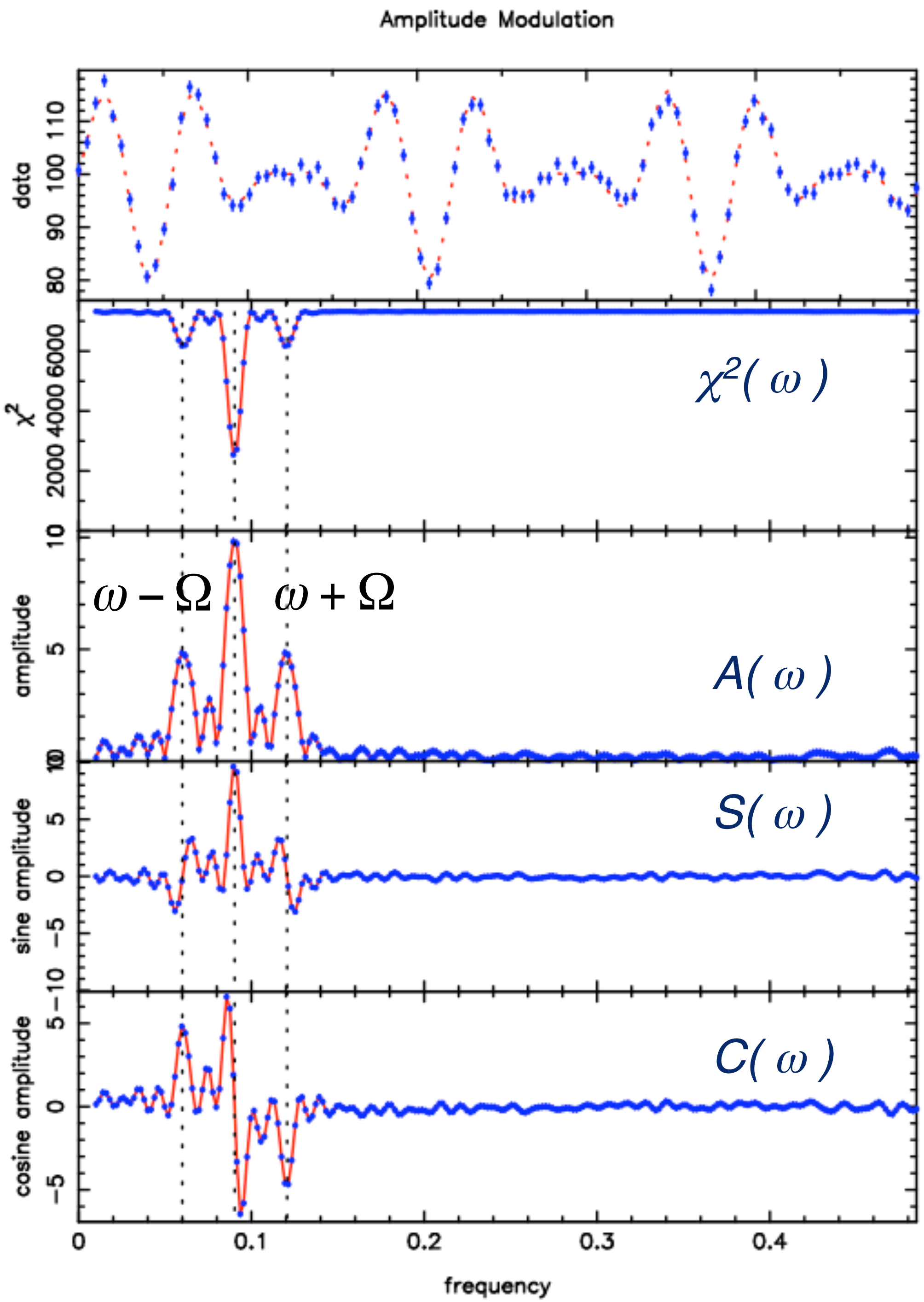
$$\Phi(t) = \alpha \sin \Omega t + \beta \cos \Omega t$$

Amplitude Modulation

Set $A_0 = 1$ and $\Phi_0 = 0$.

Oscillation frequency ω with slov
amplitude modulation at lower
frequency Ω:

$$X(t) = (1 + A\sin\Omega t + B\cos\Omega t)\sin\omega t$$
$$= \sin\omega t$$
$$+ (A/2)[\cos(\omega - \Omega)t - \cos(\omega + \Omega)t]$$
$$+ (B/2)[\sin(\omega - \Omega)t + \sin(\omega + \Omega)t]$$



Note: Sidelobes at $\omega \pm \Omega$.

Sine amplitudes in phase.

Cosine amplitudes anti-phased.

Phase Modulation

Oscillation frequency ω with slow phase modulation at lower frequency Ω:

$$X(t) = \sin(\omega t + \alpha \sin \Omega t + \beta \cos \Omega t)$$

Note: $\sin(x + \Delta x) \approx \sin x + \Delta x \cos x$:

$$X(t) \approx \sin \omega t + \left(\alpha \cos \omega t\right) \sin \Omega t + \left(\beta \cos \omega t\right) \cos \Omega t$$

$$= \sin \omega t$$

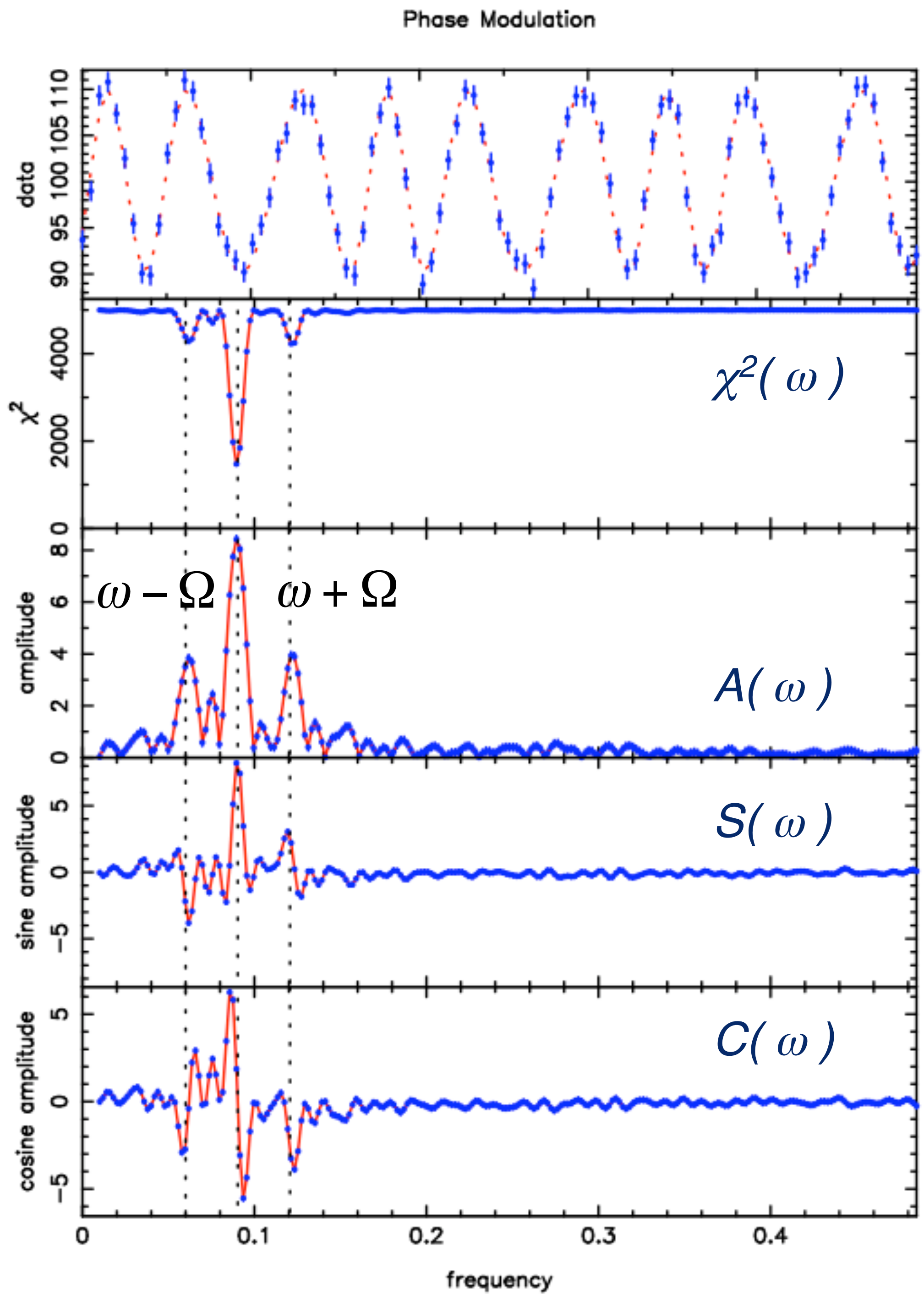
$$+ (\alpha / 2) [-\sin(\omega - \Omega) t + \sin(\omega + \Omega) t]$$

$$+ (\beta / 2) [\cos(\omega - \Omega) t + \cos(\omega + \Omega) t]$$

Again, sidelobes at $\omega \pm \Omega$ but now with

Sine amplitudes anti-phased.

Cosine amplitudes in phase



Phase relations for sidelobes

Both Amplitude and Phase Modulation:

$$X(t) = (1 + A\sin\Omega t + B\sin\Omega t)\sin(\omega t + \alpha\sin\Omega t + \beta\sin\Omega t)$$

$$= \sin\omega t \quad + \quad \frac{B-\alpha}{2}\sin(\omega-\Omega)t \quad + \quad \frac{A+\beta}{2}\cos(\omega-\Omega)t$$

$$+ \quad \frac{B+\alpha}{2}\sin(\omega+\Omega)t \quad - \quad \frac{A-\beta}{2}\cos(\omega+\Omega)t$$

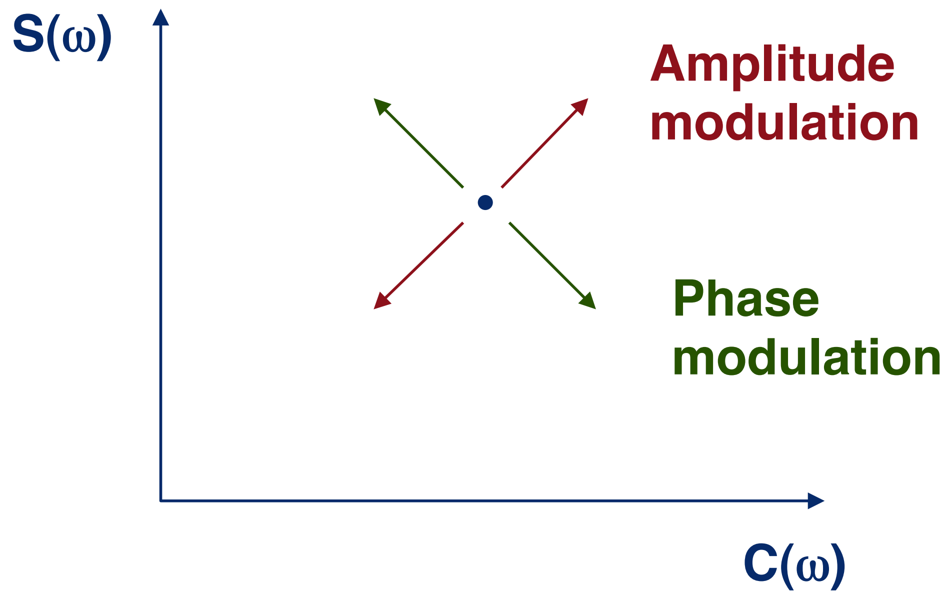
Amplitude and Phase Modulation Spectra:

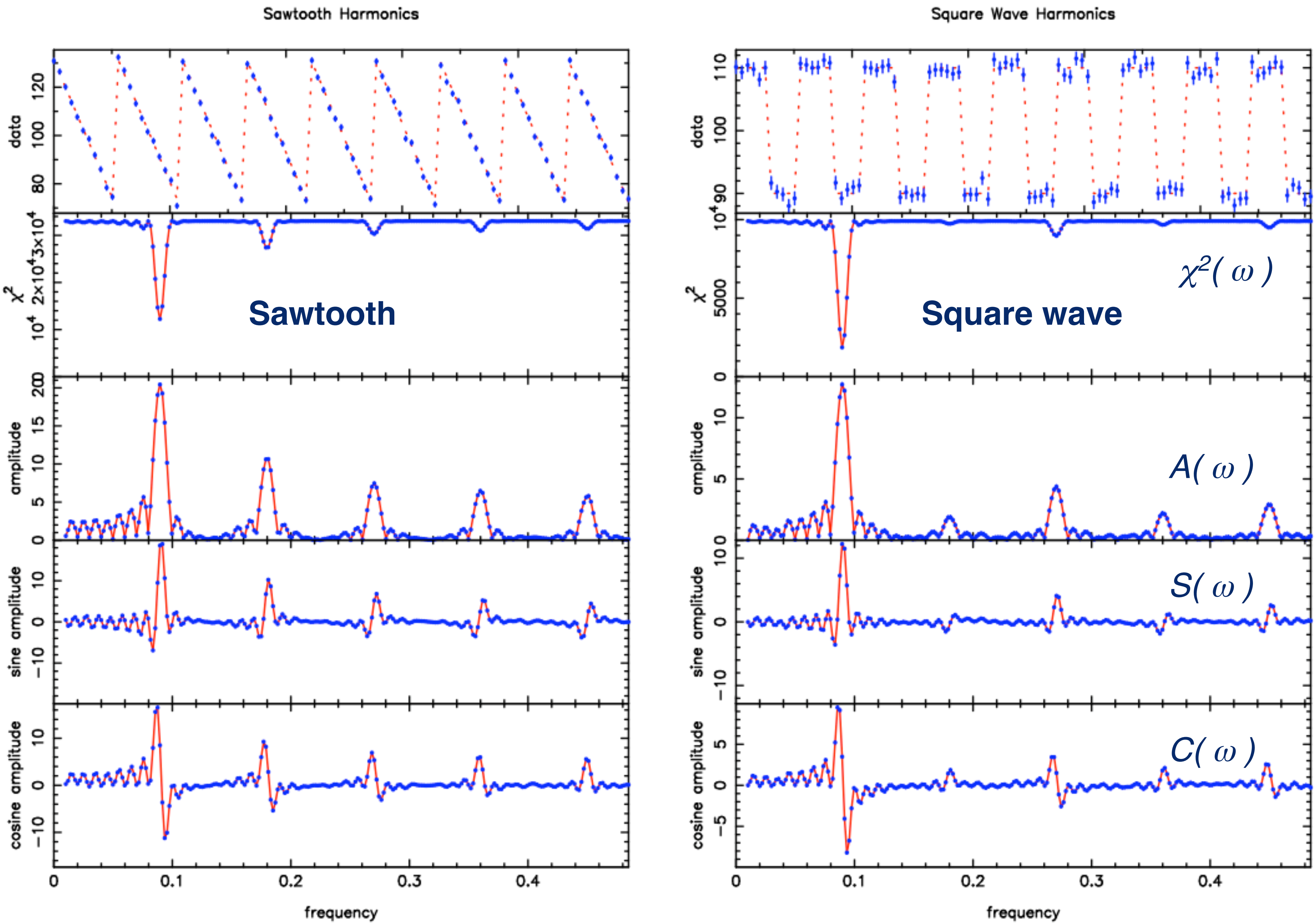
$$A(\Omega) = C(\omega - \Omega) - C(\omega + \Omega)$$

$$B(\Omega) = S(\omega - \Omega) + S(\omega + \Omega)$$

$$\alpha(\Omega) = -S(\omega - \Omega) + S(\omega + \Omega)$$

$$\beta(\Omega) = C(\omega - \Omega) + C(\omega + \Omega)$$





Sawtooth Harmonics
Square Wave Harmonics
Sawtooth
Square wave
$\chi^2(\omega)$
$A(\omega)$
$S(\omega)$
$C(\omega)$
data
χ^2
amplitude
sine amplitude
cosine amplitude
frequency

Non-sinusoidal Waveforms

- **Fundamental frequency:** ω_0
- **Harmonics** at $\omega = k\,\omega_0$, $k = 2, \ldots$ modify the **shape** of the waveform.
- Fit any shape periodic function by including amplitudes for :
  - sin($2\omega_0 t$), cos($2\omega_0 t$)
  - sin($3\omega_0 t$), cos($3\omega_0 t$)
  - etc

$$X(t) = \hat{X}_0 + \sum_{k=1}^{\infty} \left[\hat{S}_k \sin\left(k\omega t\right) + \hat{C}_k \cos\left(k\omega t\right) \right]$$

$$\hat{A}_k^2 = \hat{S}_k^2 + \hat{C}_k^2, \quad \hat{\phi}_k = \mathrm{atan2}(-\hat{S}_k, \hat{C}_k)$$

- The harmonics are approximately orthogonal (for well-sampled data with uniform phase coverage).
- Add harmonics to the model until their values become poorly determined -- Occam's razor, simplest model that fits.
- Use e.g. the BIC to decide which terms to include/omit.
- Harmonics above the Nyquist frequency will be aliased, by "folding back" across ω_{Nyq}, from ω to $\omega_{\mathrm{Nyq}} - (\omega - \omega_{\mathrm{Nyq}})$.

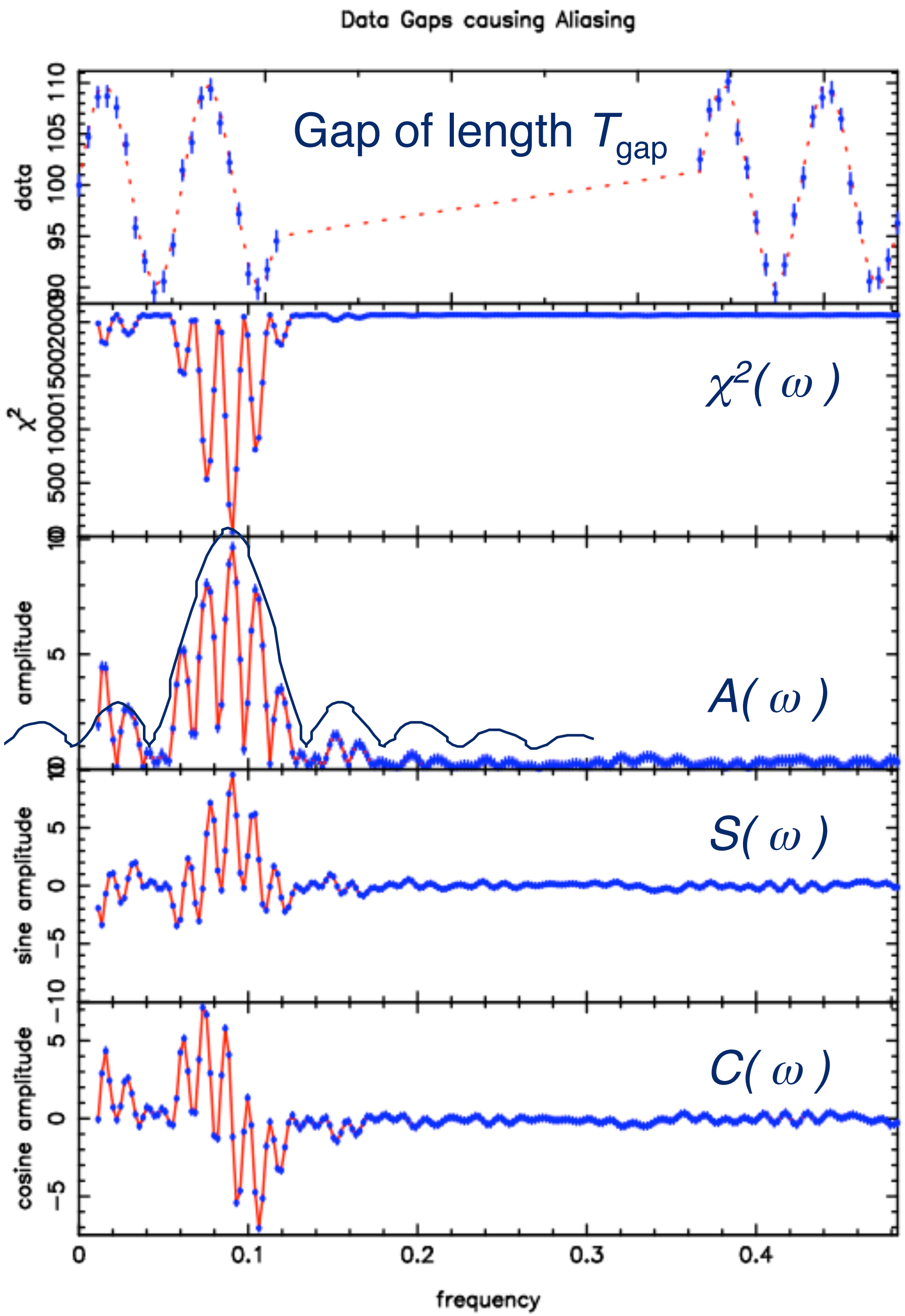
Data gaps and aliasing

Cycle-count ambiguity:
How many cycles elapse in the gap between two data segments?

- Periodogram has **sidelobes (aliases)** spaced by

$$\Delta\omega = \frac{2\pi}{T_{\text{gap}}} \qquad \Delta f = \frac{1\ \text{cycle}}{T_{gap}}$$

- Sidelobes appear within a broader **envelope** determined by duration of data segments.



Dynamic Power Spectrum

For periodic oscillations with amplitude and phase that vary with time.

Data: $X_i \pm \sigma_i$ at $t = t_i$

Model: $\mu(t) = X_0(t) + S(t)\sin(\omega t) + C(t)\cos(\omega t)$

3 Patterns: 1, $s_i = \sin(\omega t_i)$, $c_i = \cos(\omega t_i)$

Like Running Optimal Average, but including Sin and Cos amplitudes in the fit to each time window.

Iterated Optimal Scaling:

$$\hat{X}_0(t) = \frac{\sum\left(X_i - \hat{S}\,s_i - \hat{C}\,c_i\right) w_i(t)}{\sum w_i(t)},$$

$$\hat{S}(t) = \frac{\sum\left(X_i - \hat{X}_0 - \hat{C}\,c_i\right) s_i\, w_i(t)}{\sum s_i^2 w_i(t)},$$

$$\hat{C}(t) = \frac{\sum\left(X_i - \hat{X}_0 - \hat{S}\,s_i\right) c_i\, w_i(t)}{\sum c_i^2 w_i(t)},$$

$$\hat{A}^2(t) = \hat{C}^2(t) + \hat{S}^2(t)$$

$$w_i(t) = \frac{G(t - t_i)}{\sigma_i^{\,2}}$$

$$G(t) = \exp\left\{-\frac{t^2}{2\Delta^2}\right\}$$

Time-resolution set by parameter Δ.

Iterate (patterns not orthogonal).

Dynamic Power Spectrum

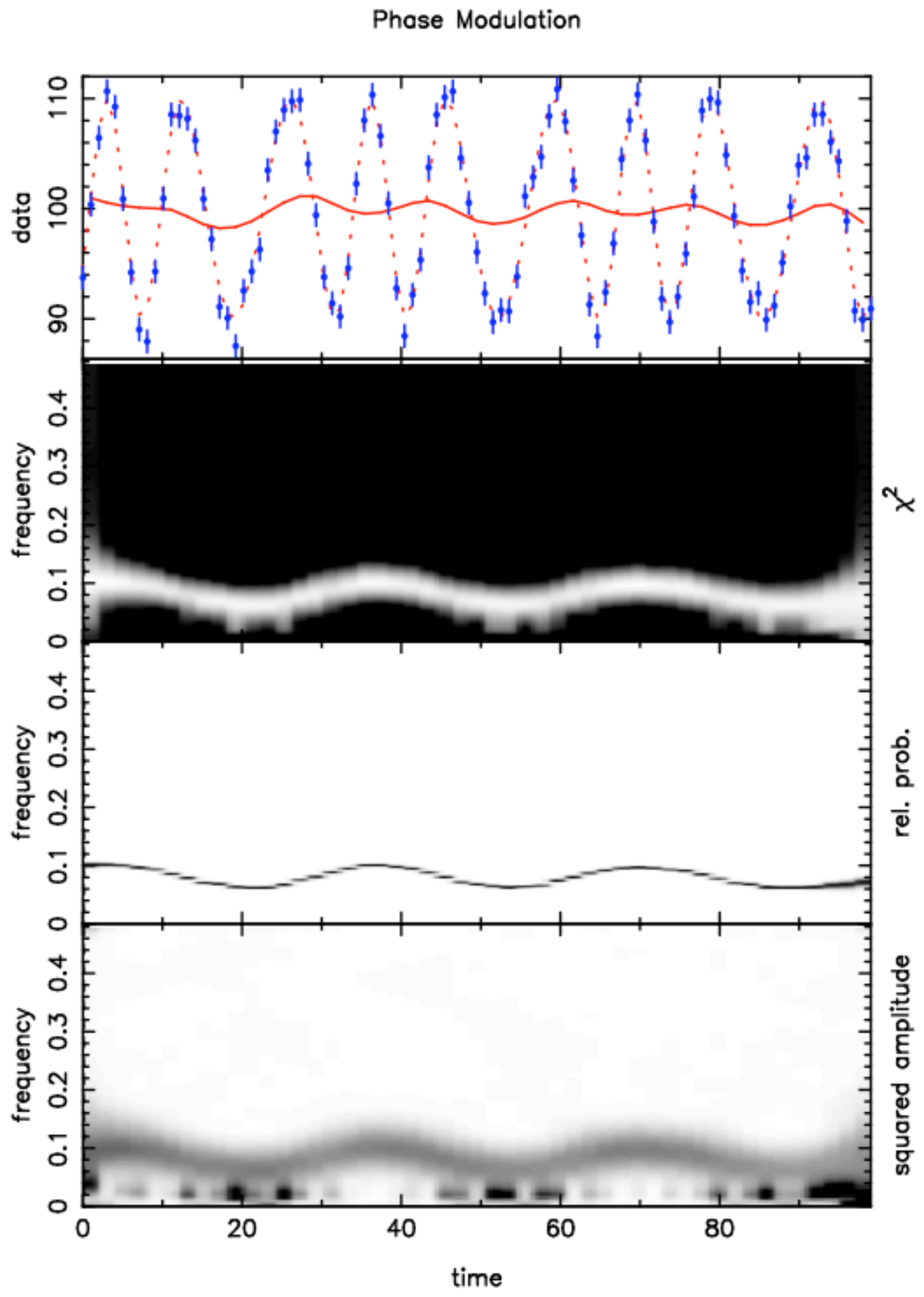
Phase modulation is equivalent to a wandering frequency.

Badness-of-Fit: $\chi^2(\omega, t)$

Probability: $P \sim exp\{-\chi^2/2\}$

Power density: $A^2(\omega, t)$

Note probability peak much sharper than power density peak.

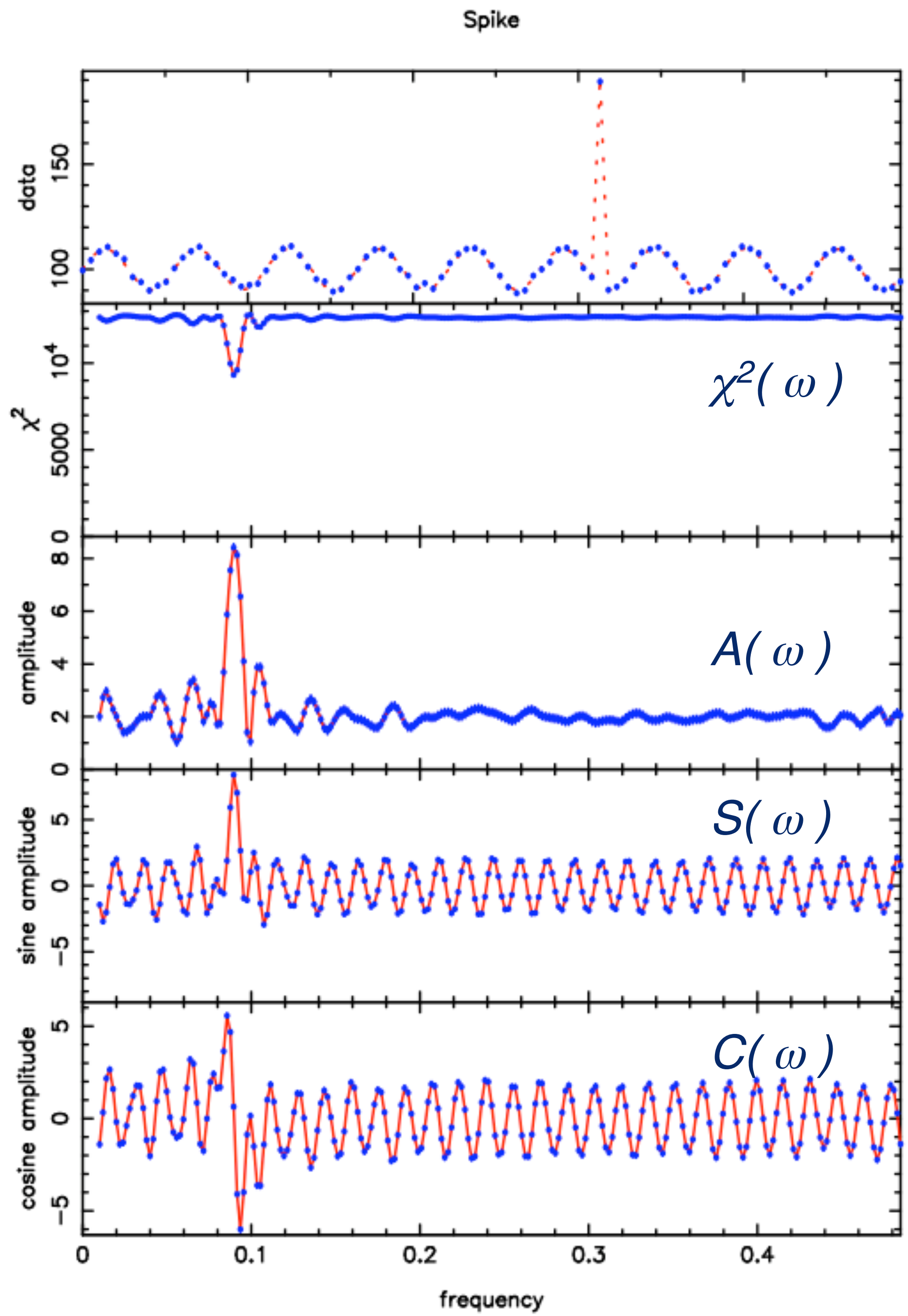


Periodogram of a sinusoid + spike

Single high value is sum of cosine curves all in phase at time t_0:

$$\delta(t - t_0) = \sum_{\omega} \cos\omega(t - t_0)\, d\omega$$

Raises the amplitude uniformly at all frequencies.

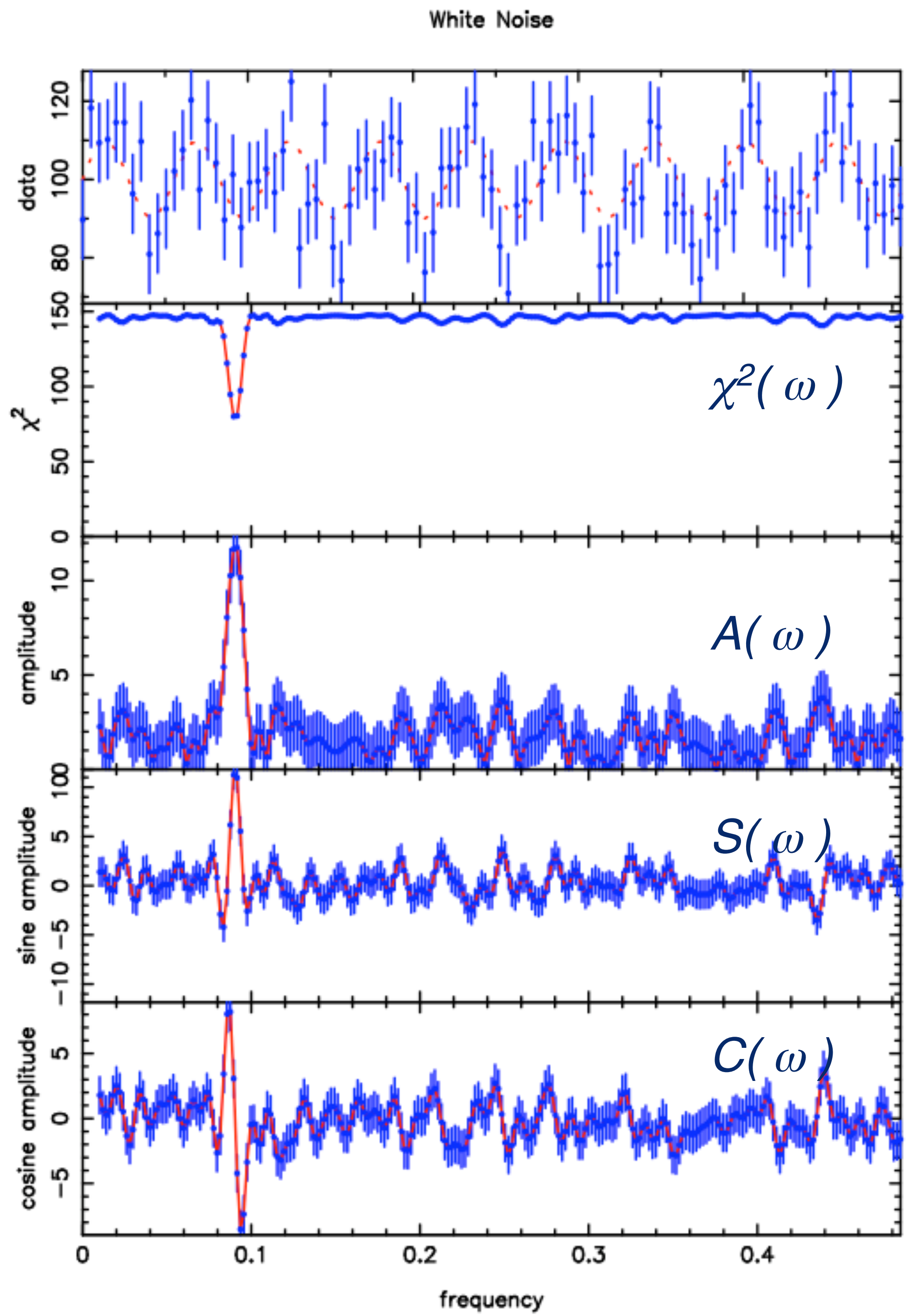


Periodogram of a sinusoid + white noise

- **White noise** can be generated by using sine curves with equal amplitude but random phases:

$$w(t) = \sum_{\omega} \cos\left(\omega\, t + \phi(\omega)\right)\, d\omega$$

- Note: Both white noise and a spike have flat periodograms.



Summary: Fourier Analysis

model: $\mu(t) = \mu_0 + \sum_k c_k \cos\omega_k t + s_k \sin\omega_k t$

even spacing: $t_i = t_0 + i\,\Delta t \qquad T = N\,\Delta t \qquad i = 1,2,\ldots,N$

Fourier frequencies: $\omega_k = k\,\Delta\omega = 2\pi/P_k \quad P_k = T/k \quad k = 0,1,\ldots,K_{\max} = N/2$

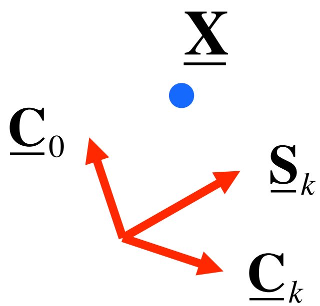
Nyquist frequency: $\omega_{Nyq} = \pi/\Delta t = 2\pi/P_{Nyq} \quad P_{Nyq} = 2\,\Delta t$

Orthogonal basis: $\underline{\mathbf{C}}_k = \cos\omega_k \underline{\mathbf{t}} \quad \underline{\mathbf{S}}_k = \sin\omega_k \underline{\mathbf{t}}$

Exact fit possible by using N parameters to fit N data points.

Model: $\underline{\mu} = \mu_o \underline{\mathbf{C}}_0 + \sum_{k=1} \left(c_k \underline{\mathbf{C}}_k + s_k \underline{\mathbf{S}}_k \right)$

Badness-of-fit: $\chi^2 = \left\| \underline{\mathbf{X}} - \underline{\mu} \right\|$



Optimal fit: $\hat{\mu}_0 = \dfrac{\underline{\mathbf{X}} \bullet \underline{\mathbf{C}}_0}{\underline{\mathbf{C}}_0 \bullet \underline{\mathbf{C}}_0} \quad \hat{c}_k = \dfrac{\underline{\mathbf{X}} \bullet \underline{\mathbf{C}}_k}{\underline{\mathbf{C}}_k \bullet \underline{\mathbf{C}}_k} \quad \hat{s}_k = \dfrac{\underline{\mathbf{X}} \bullet \underline{\mathbf{S}}_k}{\underline{\mathbf{S}}_k \bullet \underline{\mathbf{S}}_k}$

Power spectrum: $P(\omega_k) = \hat{A}_k^{\,2} \equiv \hat{c}_k^{\,2} + \hat{s}_k^{\,2}$

Decomposes lightcurve into frequency components.