Timing Analysis - Defining an Ephemeris

Timings: Observed times of a fiducial point in a periodic lightcurve, e.g. mid-eclipse.

$t_i \pm \sigma_i$

Ephemeris:

$t = t_0 + P\,E$ = predicted time

t_0 = epoch of phase 0

P = period

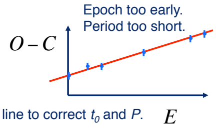
$E = n + \phi$ = cycle number + phase

$O - C$ = observed - calculated

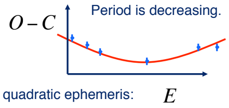
$= t_i - (t_0 + P\,E_i)$

$n_i = \mathrm{NINT}\left[\frac{t_i - t_0}{P}\right]$

phase: $\phi_i = \frac{t_i - t_0}{P} - n_i$



Fit a line to correct t_0 and P.



Fit quadratic ephemeris:

$t = t_0 + P_0\,E + B\,E^2$

$P = dt/dE = P_0 + 2\,B\,E$

$\dot{P} = dP/dt = 2\,B/P$

Sinusoidal Signals

Search a time series for a sinusoidal oscillation of unknown frequency ω:

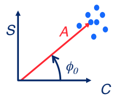
- Fit a sinusoid (scale 3 patterns):

$X(t) = X_0 + A\cos(\omega\,t + \phi_0)$

$= X_0 + C\cos\omega\,t + S\sin\omega\,t$

Amplitude: $A^2 = C^2 + S^2$

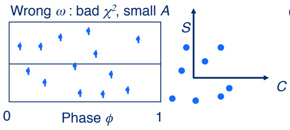
Phase at $t = 0$: $\phi_0 = \tan^{-1}(-S/C)$



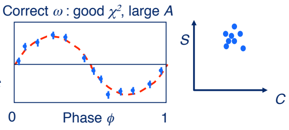
Programming hint:
Use `phi=atan2(-S,C)` if you care about which quadrant ϕ ends up in!

"Fold" data on a trial period $P = 2\pi/\omega$

Wrong ω : bad χ^2, small A



Correct ω : good χ^2, large A

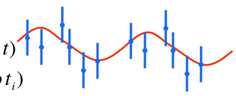


Sinusoid + Background

Data: $X_i \pm \sigma_i$ at $t = t_i$

Model: $X(t) = X_0 + S\sin(\omega\,t) + C\cos(\omega\,t)$

3 Patterns: 1, $s_i = \sin(\omega\,t_i)$, $c_i = \cos(\omega\,t_i)$



Iterated Optimal Scaling:

$$\hat{X}_0 = \frac{\sum\left(X_i - \hat{S}\,s_i - \hat{C}\,c_i\right)/\sigma_i^2}{\sum 1/\sigma_i^2}, \quad \mathrm{Var}\left[\hat{X}_0\right] = \frac{1}{\sum 1/\sigma_i^2}$$

$$\hat{S} = \frac{\sum\left(X_i - \hat{X}_0 - \hat{C}\,c_i\right)s_i/\sigma_i^2}{\sum s_i^2/\sigma_i^2}, \quad \mathrm{Var}\left[\hat{S}\right] = \frac{1}{\sum s_i^2/\sigma_i^2}$$

$$\hat{C} = \frac{\sum\left(X_i - \hat{X}_0 - \hat{S}\,s_i\right)c_i/\sigma_i^2}{\sum c_i^2/\sigma_i^2}, \quad \mathrm{Var}\left[\hat{C}\right] = \frac{1}{\sum c_i^2/\sigma_i^2}$$

Iterate (if patterns not orthogonal).

Variance formulas assume orthogonal parameters, otherwise give too small error bars.

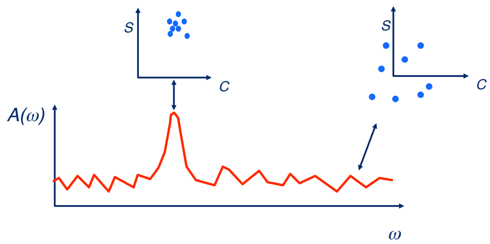
Need to use the inverse-Hessian matrix if phase coverage is not close to uniform.

Periodogram

Model is non-linear in ω (or $P = 2\pi/\omega$).

Use **grid-search**: fit sine curve for a grid of ω values.

Periodogram: plot $A(\omega)$ and/or $\chi^2(\omega)$.



Periodogram of a finite data train

Purely sinusoidal time variation.

Sampled at N regularly spaced time intervals Δt

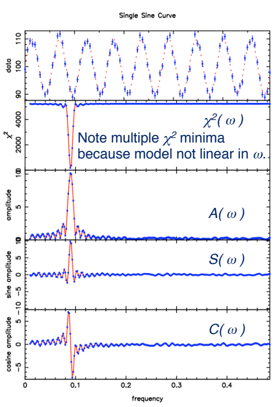
The **periodogram**:

Note χ^2 minimum and peak in A at correct ω.

Use $\Delta\chi^2 = 1$ to find $\sigma(\omega)$.

Note sidelobes and finite width of peak.

Why not a delta function?



Spectral Leakage

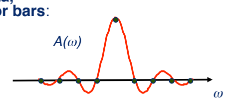
A pure sinusoid at frequency ω_0 "leaks" into adjacent frequencies ω due to the finite timespan T of the data.

$$\hat{A}(\omega) \propto \frac{\sum(\sin\omega_0\,t_i)(\sin\omega\,t_i)/\sigma_i^2}{\sum(\sin\omega\,t_i)^2/\sigma_i^2}$$

(Optimal Scaling !)

For the special case of **evenly spaced data**, at times $t_i = i\,\Delta t$, $i=1,..N$, with **equal error bars**:

$$\hat{A}(\omega) \propto \frac{\sin\pi\,x}{\pi\,x} \quad \text{where} \quad x = \frac{\omega - \omega_0}{\Delta\omega}$$

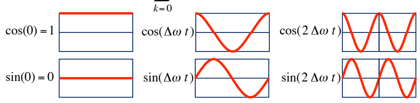


This "Sinc" function has a $1/x$ envelope and **evenly spaced zeroes** at frequency step $\Delta\omega = 2\pi/N\,\Delta t$.

De-tuning by $\Delta\omega$ gives an **orthogonal function** with 1 extra cycle per time $T = N\,\Delta t$.

Fourier Basis Functions

$$f(t) = \sum_{k=0}^{K_{max}} \left[S_k \sin(\omega_k t) + C_k \cos(\omega_k t) \right]$$



De-tuning by $\Delta\omega$ gives an orthogonal function with 1 extra cycle during time *T*.

Orthogonal for **evenly-spaced data** with **equal error bars**.

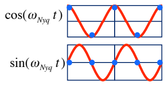
$t_i = t_0 + i\,\Delta t \quad i = 1,2,\ldots,N \qquad T = N\,\Delta t$

Fourier frequencies:

$\omega_k = k\,\Delta\omega \quad k = 0,1,\ldots,K_{max} \qquad \Delta\omega = 2\pi / T$

Nyquist frequency = 1 cycle / 2 data points

$$\omega_{Nyq} = \frac{2\pi}{2\Delta t} = \frac{N\pi}{T} = \frac{N}{2}\Delta\omega \;\Rightarrow\; K_{max} = \frac{N}{2}$$

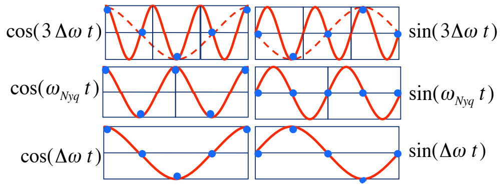


Degrees of freedom: $2(1 + K_{max}) - 2 = N$

since $\sin(\omega_0 t) = 0 \quad \sin(\omega_{Nyq} t) = 0$

Exact fit possible !

Aliasing above the Nyquist Frequency

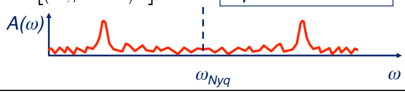


Sampled pattern is the same at $\omega_{Nyq} + k\,\Delta\omega$ and $\omega_{Nyq} - k\,\Delta\omega$.

$$\cos\left[\left(\omega_{Nyq} + k\,\Delta\omega\right) t_i\right] = \cos\left[\left(\omega_{Nyq} - k\,\Delta\omega\right) t_i\right]$$

$$\sin\left[\left(\omega_{Nyq} + k\,\Delta\omega\right) t_i\right] = -\sin\left[\left(\omega_{Nyq} - k\,\Delta\omega\right) t_i\right]$$

Frequencies above Nyquist frequency duplicate those below.



Periodogram

Pure sinusoid signal.

Sampled at *N* regularly spaced time intervals Δt

The **periodogram**:

Note χ^2 minimum and peak in *A* at correct ω.

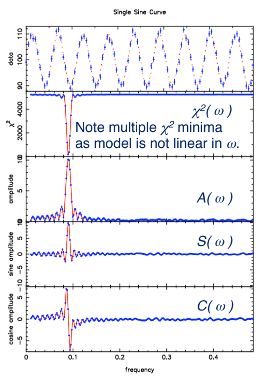
Use $\Delta\chi^2 = 1$ to find $\sigma(\omega)$.

Sidelobe spacing:

$\Delta\omega = 2\pi / T = 2\pi / N\,\Delta T$

Nyquist frequency:

$\omega_N = (N/2)\,\Delta\omega = \pi N / T = \pi / \Delta T$



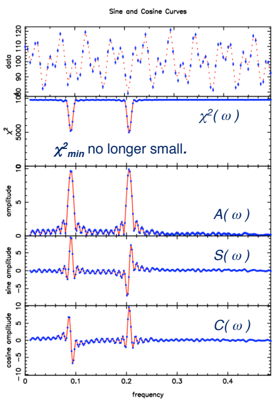
Widely spaced frequencies

Sum of sine and cosine curves at well-separated frequencies.

Periodogram shows two well separated peaks.

χ^2_{min} is high, but can still use $\Delta\chi^2 = 1$ to find $\sigma(\omega)$.

(This is how we find multiple planets in Doppler data)



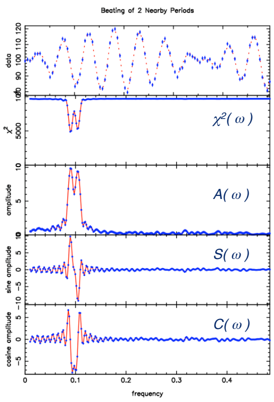
Closely spaced frequencies

Wave trains drift in and out of phase.

Constructive and destructive interference produces "**beating**" in the light curve.

Beat frequency $\omega_B = |\omega_1 - \omega_2|$

Peaks overlap in periodogram.



"Pre-whitening"

Separate closely-spaced frequencies by ***"pre-whitening"*** the data.

Fit and subtract strongest period, then fit the next, etc.

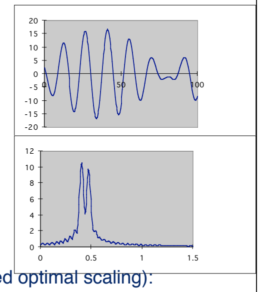
Subtract $A_1 \sin(\omega_1 t - \phi_1)$

Fit $A_2 \sin(\omega_2 t - \phi_2)$ to residuals

Subtract $A_2 \sin(\omega_2 t - \phi_2)$

Fit $A_1 \sin(\omega_1 t - \phi_1)$ to residuals

Iterate to convergence



Fits a 7-parameter model (e.g. by iterated optimal scaling):

$$X(t) = X_0 + A_1 \sin(\omega_1 t + \phi_1) + A_2 \sin(\omega_2 t + \phi_2)$$
$$= X_0 + S_1 \sin(\omega_1 t) + C_1 \cos(\omega_1 t) + S_2 \sin(\omega_2 t) + C_2 \cos(\omega_2 t)$$

2 non-linear params: ω_1, ω_2, 5 linear params: X_0, S_1, C_1, S_2, C_2