

Review: Fit a line to N data points

Correlated parameters:
 $y = a x + b$

Orthogonal parameters:

$$y = a(x - \hat{x}) + b \quad \hat{x} = \frac{\sum x_i / \sigma_i^2}{\sum 1 / \sigma_i^2}$$

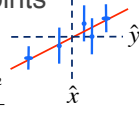
For intercept b , **optimal average:**

$$\hat{b} = \frac{\sum y_i / \sigma_i^2}{\sum 1 / \sigma_i^2}, \quad \text{Var}[\hat{b}] = \frac{1}{\sum 1 / \sigma_i^2}$$

For slope a , **optimal scaling:**

$$\hat{a} = \frac{\sum y_i (x_i - \hat{x}) / \sigma_i^2}{\sum (x_i - \hat{x})^2 / \sigma_i^2}, \quad \text{Var}[\hat{a}] = \frac{1}{\sum (x_i - \hat{x})^2 / \sigma_i^2}$$

No need to iterate. (Why?)



Fit a line => fit 2 patterns => fit M patterns

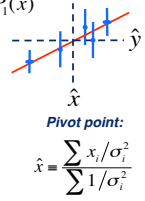
Model: $y = b + a(x - \hat{x}) = \alpha_0 P_0(x) + \alpha_1 P_1(x)$

Patterns: $P_0(x) = 1 \quad P_1(x) = (x - \hat{x})$

Optimal Scaling:

$$\hat{\alpha}_0 = \frac{\sum (y_i - \hat{\alpha}_1 P_1(x_i)) P_0(x_i) / \sigma_i^2}{\sum P_0^2(x_i) / \sigma_i^2}, \quad \text{Var}[\hat{\alpha}_0] = \frac{1}{\sum P_0^2(x_i) / \sigma_i^2}$$

$$\hat{\alpha}_1 = \frac{\sum (y_i - \hat{\alpha}_0 P_0(x_i)) P_1(x_i) / \sigma_i^2}{\sum P_1^2(x_i) / \sigma_i^2}, \quad \text{Var}[\hat{\alpha}_1] = \frac{1}{\sum P_1^2(x_i) / \sigma_i^2}$$



Iterate (if patterns not orthogonal).

LINEAR REGRESSION:
 Generalise model to M patterns:

$$y = \sum_{k=1}^M \alpha_k P_k(x)$$

Iterated Optimal Scaling: simple algorithm, easy to code, often adequate.

Example: Sine Curve + Background

Data: $X_i \pm \sigma_i$ at $t = t_i$

Model: $X(t) = A + S \sin(\omega t) + C \cos(\omega t)$

3 Patterns: 1, $s_i = \sin(\omega t_i)$, $c_i = \cos(\omega t_i)$

Iterated Optimal Scaling:

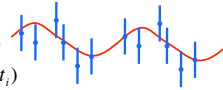
$$\hat{A} = \frac{\sum (X_i - \hat{S} s_i - \hat{C} c_i) / \sigma_i^2}{\sum 1 / \sigma_i^2}, \quad \text{Var}[\hat{A}] = \frac{1}{\sum 1 / \sigma_i^2}$$

$$\hat{S} = \frac{\sum (X_i - \hat{A} - \hat{C} c_i) s_i / \sigma_i^2}{\sum s_i^2 / \sigma_i^2}, \quad \text{Var}[\hat{S}] = \frac{1}{\sum s_i^2 / \sigma_i^2}$$

$$\hat{C} = \frac{\sum (X_i - \hat{A} - \hat{S} s_i) c_i / \sigma_i^2}{\sum c_i^2 / \sigma_i^2}, \quad \text{Var}[\hat{C}] = \frac{1}{\sum c_i^2 / \sigma_i^2}$$

Iterate (if patterns not orthogonal).

Variance formulas assume orthogonal parameters, otherwise give error bars too small. Use inverse of Hessian matrix (see later).



$$\chi^2 = \sum_{i=1}^N \left(\frac{y_i - (ax_i + b)}{\sigma_i} \right)^2$$

$$0 = \frac{\partial \chi^2}{\partial a} = -2 \sum x(y - ax - b) / \sigma^2$$

$$0 = \frac{\partial \chi^2}{\partial b} = -2 \sum (y - ax - b) / \sigma^2$$

The Normal Equations:

$$a \sum x^2 / \sigma^2 + b \sum x / \sigma^2 = \sum xy / \sigma^2$$

$$a \sum x / \sigma^2 + b \sum 1 / \sigma^2 = \sum y / \sigma^2$$

Matrix form:

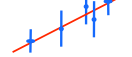
$$\begin{pmatrix} \sum x^2 / \sigma^2 & \sum x / \sigma^2 \\ \sum x / \sigma^2 & \sum 1 / \sigma^2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \sum xy / \sigma^2 \\ \sum y / \sigma^2 \end{pmatrix}$$

$$\underline{H} \underline{\alpha} = \underline{c}(y) \quad (\underline{H} = \text{Hessian matrix})$$

$$\text{Solution: } \underline{\alpha} = \underline{H}^{-1} \underline{c}(y)$$

χ^2 analysis of the straight line fit

$y = ax + b$



Normal Equations: $\underline{H} \underline{\alpha} = \underline{c}(y)$

$$\begin{pmatrix} \sum x^2 / \sigma^2 & \sum x / \sigma^2 \\ \sum x / \sigma^2 & \sum 1 / \sigma^2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \sum xy / \sigma^2 \\ \sum y / \sigma^2 \end{pmatrix}$$

Solution: $\underline{\alpha} = \underline{H}^{-1} \underline{c}(y)$

$$\begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{\Delta} \begin{pmatrix} \sum 1 / \sigma^2 & -\sum x / \sigma^2 \\ -\sum x / \sigma^2 & \sum x^2 / \sigma^2 \end{pmatrix} \begin{pmatrix} \sum xy / \sigma^2 \\ \sum y / \sigma^2 \end{pmatrix}$$

Hessian Determinant: $\Delta = (\sum 1 / \sigma^2)(\sum x^2 / \sigma^2) - (\sum x / \sigma^2)^2$

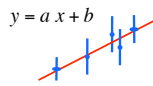
Orthogonal basis: $x \Rightarrow (x - \hat{x}) \quad \hat{x} = (\sum x / \sigma^2) / (\sum 1 / \sigma^2)$

$$\sum (x - \hat{x}) / \sigma^2 = 0, \quad \Delta = (\sum 1 / \sigma^2)(\sum (x - \hat{x})^2 / \sigma^2)$$

$$\begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{\Delta} \begin{pmatrix} \sum 1 / \sigma^2 & 0 \\ 0 & \sum (x - \hat{x})^2 / \sigma^2 \end{pmatrix} \begin{pmatrix} \sum (x - \hat{x}) y / \sigma^2 \\ \sum y / \sigma^2 \end{pmatrix} \quad (\text{Diagonal Hessian Matrix})$$

$$\hat{a} = \frac{\sum (x - \hat{x}) y / \sigma^2}{\sum (x - \hat{x})^2 / \sigma^2} \quad \hat{b} = \frac{\sum y / \sigma^2}{\sum 1 / \sigma^2} \quad (\text{same as Optimal Scaling})$$

χ^2 analysis of the straight line fit



The Hessian Matrix

$$H_{jk} = \frac{1}{2} \frac{\partial^2 \chi^2}{\partial a_j \partial a_k}$$

$$\chi^2 = \sum_{i=1}^N \left(\frac{y_i - (ax_i + b)}{\sigma_i} \right)^2$$

$$\frac{\partial \chi^2}{\partial a} = -2 \sum x(y - ax - b) / \sigma^2$$

$$\frac{\partial \chi^2}{\partial b} = -2 \sum (y - ax - b) / \sigma^2$$

• Example: $y = ax + b$.

$$\frac{\partial^2 \chi^2}{\partial a^2} = 2 \sum x_i^2 / \sigma_i^2 \quad \frac{\partial^2 \chi^2}{\partial a \partial b} = 2 \sum x_i / \sigma_i^2$$

$$\frac{\partial^2 \chi^2}{\partial b^2} = 2 \sum 1 / \sigma_i^2, \quad \text{so } H = \begin{bmatrix} \sum x_i^2 / \sigma_i^2 & \sum x_i / \sigma_i^2 \\ \sum x_i / \sigma_i^2 & \sum 1 / \sigma_i^2 \end{bmatrix}$$

For linear models, Hessian matrix is independent of the parameters, and χ^2 surface is parabolic.

Parameter Uncertainties

Quadratic approximation to χ^2 surface uses Hessian matrix :

$$\chi^2(\alpha) = \chi^2(\hat{\alpha}) + \sum_{jk} (\alpha_j - \hat{\alpha}_j) H_{jk} (\alpha_k - \hat{\alpha}_k) + \dots \quad H_{jk} = \frac{1}{2} \frac{\partial^2 \chi^2}{\partial \alpha_j \partial \alpha_k}$$

For linear models, Hessian matrix is independent of the parameters, and χ^2 surface is parabolic.

For a one-parameter fit:

$$\text{if } \hat{\alpha} \text{ minimizes } \chi^2, \text{ then } \text{Var}(\hat{\alpha}) = \frac{2}{\partial^2 \chi^2 / \partial \alpha^2}.$$

For a multi-parameter fit the covariance of any pair of parameters is an element of the inverse-Hessian matrix:

$$\text{Cov}(a_k, a_j) = \left[\frac{1}{2} \frac{\partial^2 \chi^2}{\partial \alpha_k \partial \alpha_j} \right]^{-1} = [H^{-1}]_{kj}$$

Principal Axes of χ^2 Ellipsoid

The **eigenvectors** of H define the **principal axes** of the χ^2 ellipsoid. Equivalent to **rotating** the coordinate system in parameter space.

$$y = ax + b$$

$$= (\bar{a} \cos \theta + \bar{b} \sin \theta) x + (\bar{b} \cos \theta - \bar{a} \sin \theta)$$

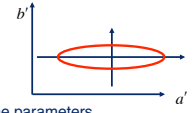
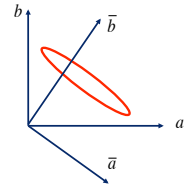
Can also diagonalise H by :

$$ax + b \rightarrow a'(x - \hat{x}) + b'$$

This **shears** the parameter space, giving

$$H = \begin{bmatrix} \sum_i (x_i - \hat{x})^2 / \sigma_i^2 & 0 \\ 0 & \sum_i 1 / \sigma_i^2 \end{bmatrix}$$

Diagonalising the Hessian matrix orthogonalises the parameters.



Scale M Patterns: General Linear Regression

$$\text{Linear Model: } y(x) = a_1 P_1(x) + a_2 P_2(x) + \dots = \sum_k a_k P_k(x)$$

Example: Polynomial: $y(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_{M-1} x^{M-1}$

$$\chi^2 = \sum_{i=1}^N \left[\frac{y_i - y(x_i)}{\sigma_i} \right]^2 = \sum_{i=1}^N \frac{1}{\sigma_i^2} \left(y_i - \sum_j a_j P_j(x_i) \right)^2$$

Normal Equations:

$$0 = \frac{\partial \chi^2}{\partial a_k} = -2 \sum_i \left(y_i - \sum_j a_j P_j(x_i) \right) \frac{P_k(x_i)}{\sigma_i^2} \quad k=1 \dots M$$

$$\sum_j \left(\sum_i \frac{P_j P_{ki}}{\sigma_i^2} \right) (a_j) = \sum_i \frac{y_i P_{ki}}{\sigma_i^2} \quad P_{ki} = P_k(x_i)$$

$$\sum_j H_{jk} a_j = c_k(y)$$

Principal Axes for general Linear Models

- In the general linear case where we fit M functions P_k with scale factors a_k :

$$y(x) = \sum_{k=1}^M a_k P_k(x)$$

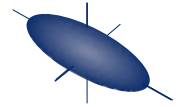
- The $(M \times M)$ Hessian matrix has elements:

$$H_{jk} = \frac{1}{2} \frac{\partial^2 \chi^2}{\partial a_j \partial a_k} = \sum_{i=1}^N \frac{P_j(x_i) P_k(x_i)}{\sigma_i^2}$$

- Normal equations (M equations for M unknowns):

$$H_{jk} a_k = c_j \quad \text{where } c_j = \sum_{i=1}^N \frac{y_i P_j(x_i)}{\sigma_i^2}$$

- This gives M -dimensional ellipsoidal surfaces of constant χ^2 whose principal axes are the M eigenvectors of the Hessian matrix H .
- Use standard matrix methods to find linear combinations of P_j that diagonalise H .



Linear vs Non-Linear Models

$$\text{Linear Model: } y(x) = \sum_k \alpha_k P_k(x) \quad M \text{ scale parameters } \alpha_k$$

$$H_{jk} = \frac{1}{2} \frac{\partial^2 \chi^2}{\partial \alpha_j \partial \alpha_k} = \sum_{i=1}^N \frac{P_j(x_i) P_k(x_i)}{\sigma_i^2}$$

Elliptical χ^2 contours, unique solution by linear regression (matrix inversion).

Non-Linear Models:

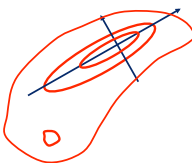
$$\text{power-law: } y = A x^B$$

$$\text{blackbody: } f_\nu = \Omega B_\nu(\lambda, T)$$

$$H_{jk} = \frac{1}{2} \frac{\partial^2 \chi^2}{\partial \alpha_j \partial \alpha_k} \text{ depends on parameters}$$

$$\chi^2(\alpha) = \chi^2(\hat{\alpha}) + \sum_{jk} (\alpha_j - \hat{\alpha}_j) H_{jk} (\alpha_k - \hat{\alpha}_k) + \dots$$

Banana-shaped contours, multiple local minima, require **iterative methods**.



Method 1: Linearise the Non-Linear Model

Linearisation: use local linear approximation to the model, giving a quadratic approximation to χ^2 surface. Solve by linear regression, then iterate.

A and B are scale parameters.

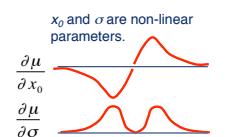
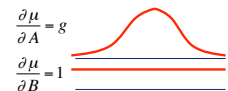
Example: gaussian peak + background :

$$\mu = Ag + B \quad g = e^{-x^2/2} \quad \eta = \frac{x - x_0}{\sigma}$$

$$\Delta \mu = \Delta A \frac{\partial \mu}{\partial A} + \Delta B \frac{\partial \mu}{\partial B} + \Delta x_0 \frac{\partial \mu}{\partial x_0} + \Delta \sigma \frac{\partial \mu}{\partial \sigma}$$

$$\frac{\partial \mu}{\partial A} = g \quad \frac{\partial \mu}{\partial x_0} = A g \eta / \sigma$$

$$\frac{\partial \mu}{\partial B} = 1 \quad \frac{\partial \mu}{\partial \sigma} = A g \eta^2 / \sigma$$



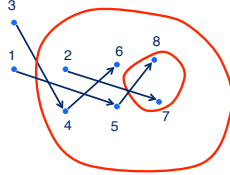
Guess x_0 and σ , fit linear parameters A and B , evaluate derivatives, adjust x_0 and σ using linear approximation, iterate.

Method 2: Amoeba (Downhill Simplex)

Amoeba (downhill simplex)

Simplex = cluster of $M+1$ points in the M -dimensional parameter space.

1. Evaluate χ^2 at each node.
2. Pick node with highest χ^2 , move it on a line thru the centroid of the other M nodes, using simple rules to find new place with lower χ^2 .
3. Repeat until converged.



Amoeba requires no derivatives :)

Amoeba "crawls" downhill, adjusting shape to match the χ^2 landscape, then shrinks down onto a local minimum.

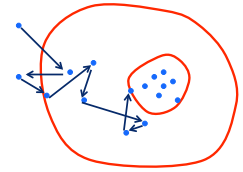
See Numerical Recipes for full description.

Method 3: Markov Chain Monte Carlo (MCMC)

1. Start with $M+1$ points in the M -dimensional parameter space.
2. Evaluate α_i for each parameter (and covariance matrix) from last n points.
3. Take a **random step**, e.g. using a Gaussian random number with same α_i (and covariances) as recent points.

$$\Delta\alpha_i \sim G(0, \sigma_i^2)$$

4. Evaluate $\Delta\chi^2 = \chi^2_{\text{new}} - \chi^2_{\text{old}}$ and keep the step with probability $P = \min\left[1, \exp(-\Delta\chi^2 / 2)\right]$
5. Iterate steps 2-4 until "convergence".



MCMC requires no derivatives :)

MCMC generates a "chain" of points tending to move downhill, then settling into a pattern matching the **posterior distribution** of the parameters.

Can escape from local minima.

Can also include prior distributions on the parameters.

Example: MCMC fit of exoplanet model to transit lightcurves and radial velocity curve data.

