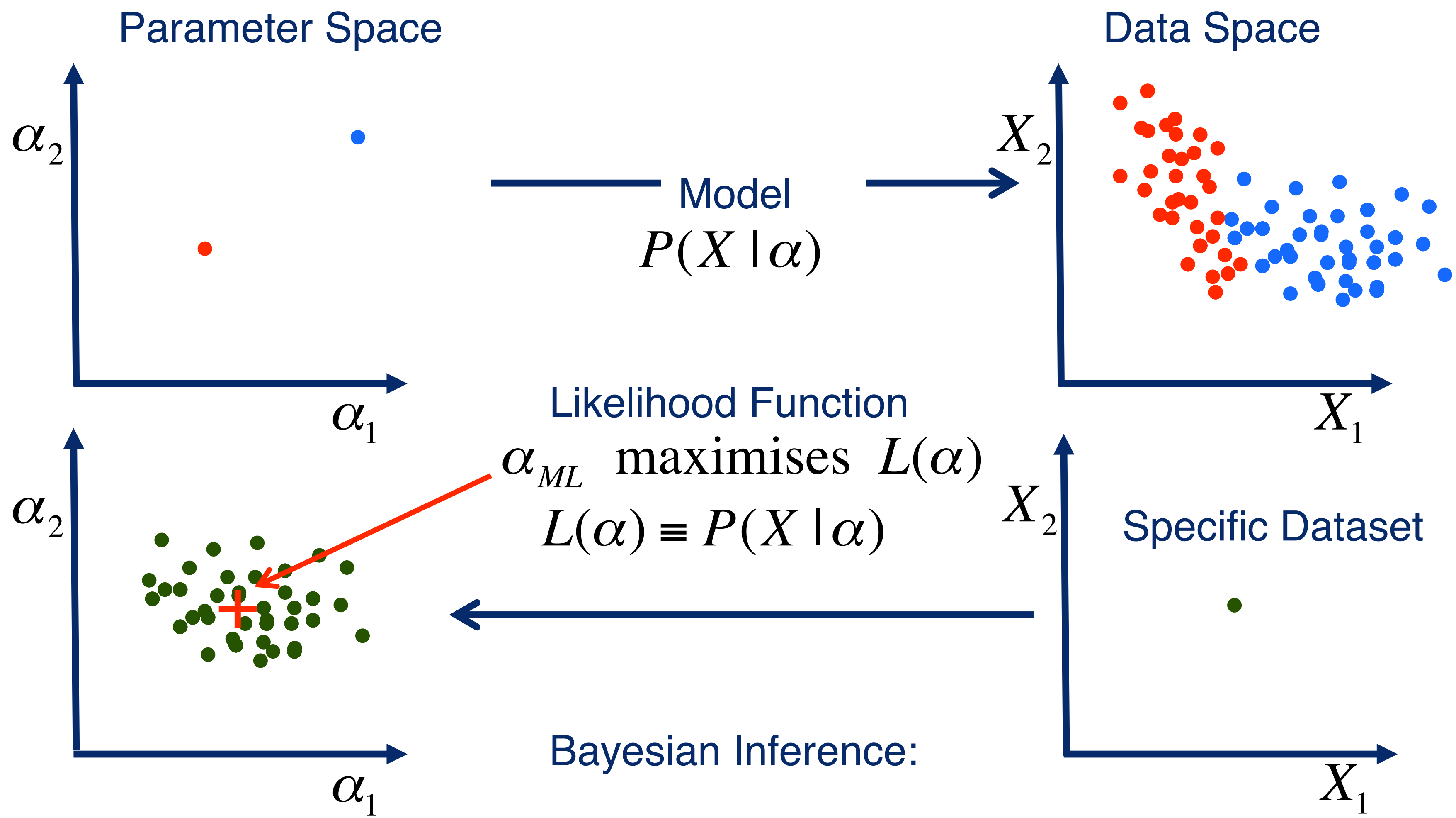
Review: Max Likelihood and Bayesian Inference



Bayesian Inference:

$$P(\alpha \mid X) = \frac{P(X \mid \alpha)\, P(\alpha)}{\int P(X \mid \alpha)\, P(\alpha)\, \mathrm{d}\alpha} \propto L(\alpha)\, P(\alpha)$$

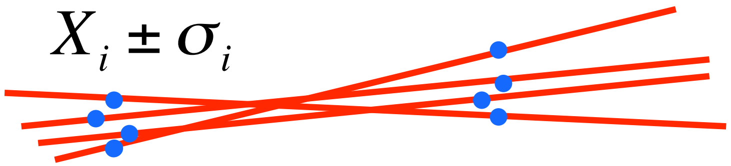
Posterior Probability

Likelihood modifies the Prior.

Monte-Carlo Error Propagation

1. Create **fake datasets**.

1a. “Jiggle” the data points (using Gaussian random numbers).
* Requires good error bars.



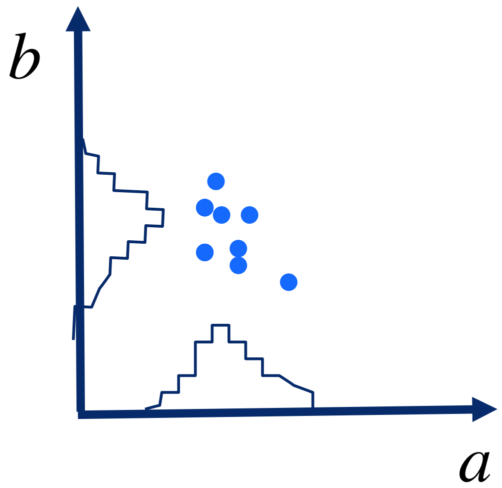
1b. (and/or) “**Bootstrap**” samples:

Pick N data points at random, with replacement (some points omitted, some repeated).
* Requires more data than parameters ($N > M$).
* Works with no error bars available.

2. Fit the model to each “fake” dataset.

$$\langle X_i \rangle = a\, t_i + b$$

3. Observe how the best-fit parameter values “dance”.



4. Accumulate histograms approximating the parameter probability distributions.

5. Compute mean, median, variance, etc. of the parameters, or **any function of the parameters**.

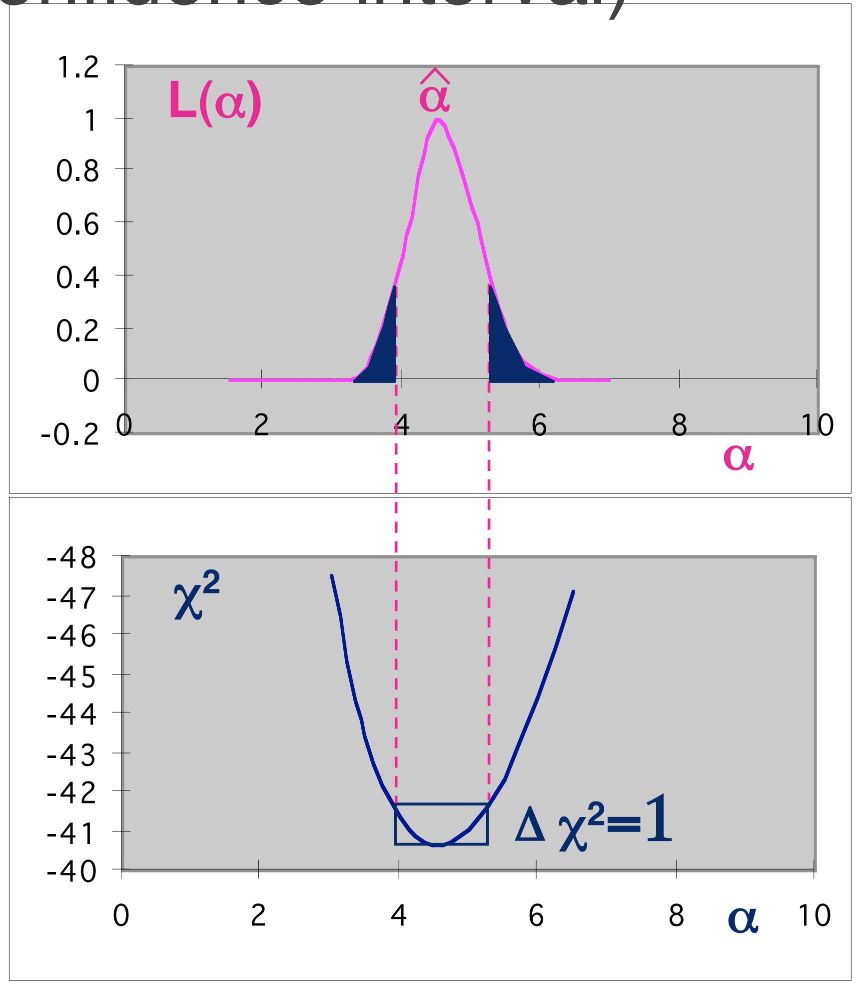
Confidence interval on a single parameter
(1-parameter, k-sigma confidence interval)

The **1-σ confidence interval** on α includes 68% of the area under the likelihood function:

$$L(\alpha) \equiv P(X \mid \alpha) \propto \frac{e^{-\chi^2/2}}{\prod_i \sigma_i}$$

or posterior probability distribution, for non-uniform prior $P(\alpha)$:

$$P(\alpha \mid X) \propto L(\alpha)\, P(\alpha)$$



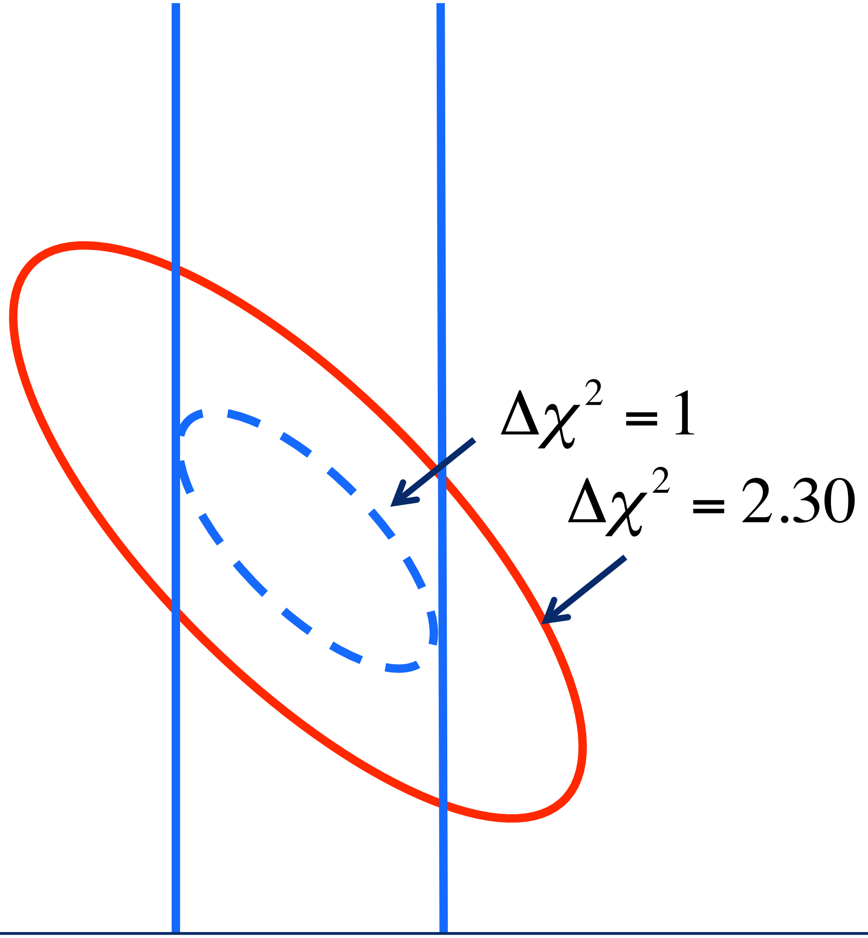
For a *k-σ* (1-parameter) confidence interval, use *$\Delta\chi^2 = k^2$*,

$\Delta\chi^2 = 1$ for *1-σ*, 68% probability
$\Delta\chi^2 = 4$ for *2-σ*, 95.4% probability
$\Delta\chi^2 = 9$ for *3-σ*, 99.73% probability …

Generalise:

χ^2 => $-2 \ln(L(\alpha)\, P(\alpha))$

2-parameter 1-sigma Confidence Region



If *Y* is a "**nuisance parameter**", use the **1-parameter 1-sigma confidence interval** in *X*, tangent to the $\Delta\chi^2 = 1$ contour in *(X, Y)*.
The interval encloses 68% probability.

If both *X* and *Y* are of interest, use the **2-parameter 1-sigma confidence region**, the $\Delta\chi^2 = 2.30$ contour in *(X, Y)*.
The contour encloses 68% probability.

Use *-2 ln(L x P)* instead of χ^2, if needed.

$$L(\alpha) \equiv P(X \mid \alpha) \propto \frac{e^{-\chi^2/2}}{\prod_i \sigma_i}$$

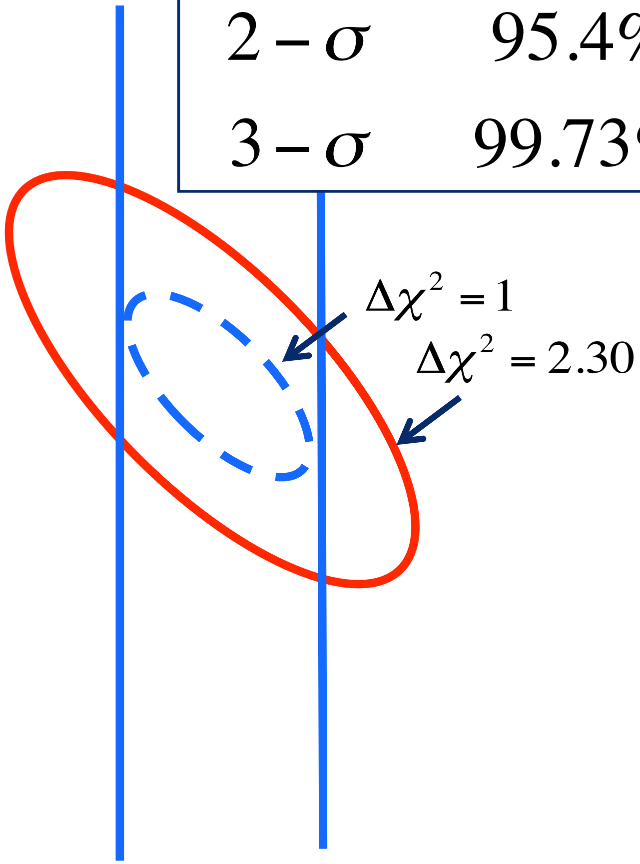
Note: Contour enclosing 68% probability must be **wider** than the 1-parameter confidence interval.

Why?

M - parameter *k* - σ Confidence Regions

$\Delta\chi^2$ thresholds for M-parameter *k*-σ Confidence Regions

	Prob	$M=1$	2	3	4
$1-\sigma$	68%	1	2.30	3.53	4.72
$2-\sigma$	95.4%	4	6.17	8.02	9.70
$3-\sigma$	99.73%	9	11.8	14.2	16.3



The ***M*-parameter confidence region** is enclosed by the $\Delta\chi^2$ surface enclosing the desired probability.

All **nuisance parameters must be re-fitted** (or integrated over) for each set of fixed values for the *M* parameters in the sub-space of interest.

The $\Delta\chi^2$ in the *M*-parameter sub-space has a χ^2_M distribution.

Example: Estimate both μ and σ

$$L(\mu,\sigma) \equiv P(X \mid \mu,\sigma) = \frac{e^{-\chi^2/2}}{(2\pi)^{N/2}\sigma^N}$$

$$-2\ln L = \sum_{i=1}^{N}\left(\frac{X_i-\mu}{\sigma}\right)^2 + 2N\ln\sigma + \text{const}$$

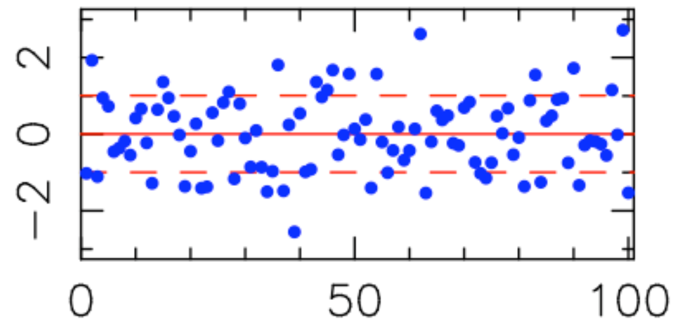
$$0 = \frac{\partial}{\partial\mu}\left[-2\ln L\right] = -2\sum_{i=1}^{N}\frac{X_i-\mu}{\sigma^2}$$

$$0 = \frac{\partial}{\partial\sigma}\left[-2\ln L\right] = -2\sum_{i=1}^{N}\frac{(X_i-\mu)^2}{\sigma^3} + \frac{2N}{\sigma}$$

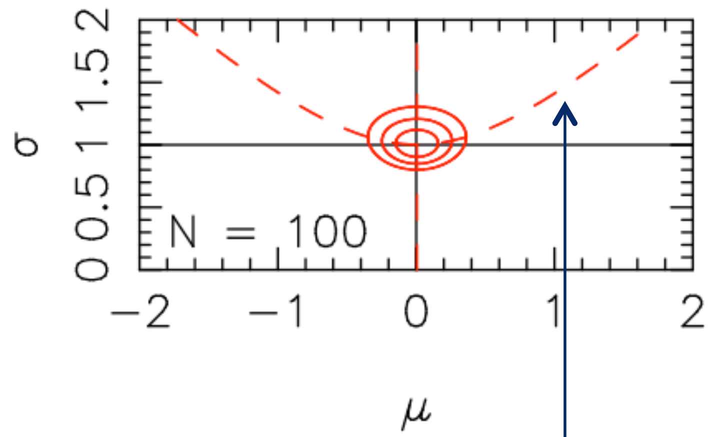
$$\mu_{ML} = \frac{1}{N}\sum_i X_i \qquad (\sigma_{ML})^2 = \frac{1}{N}\sum_i\left(X_i-\mu_{ML}\right)^2$$

Posterior ∝ Likelihood × Prior

$$P(\mu,\sigma \mid X) \propto L(\mu,\sigma)\, P(\mu,\sigma)$$

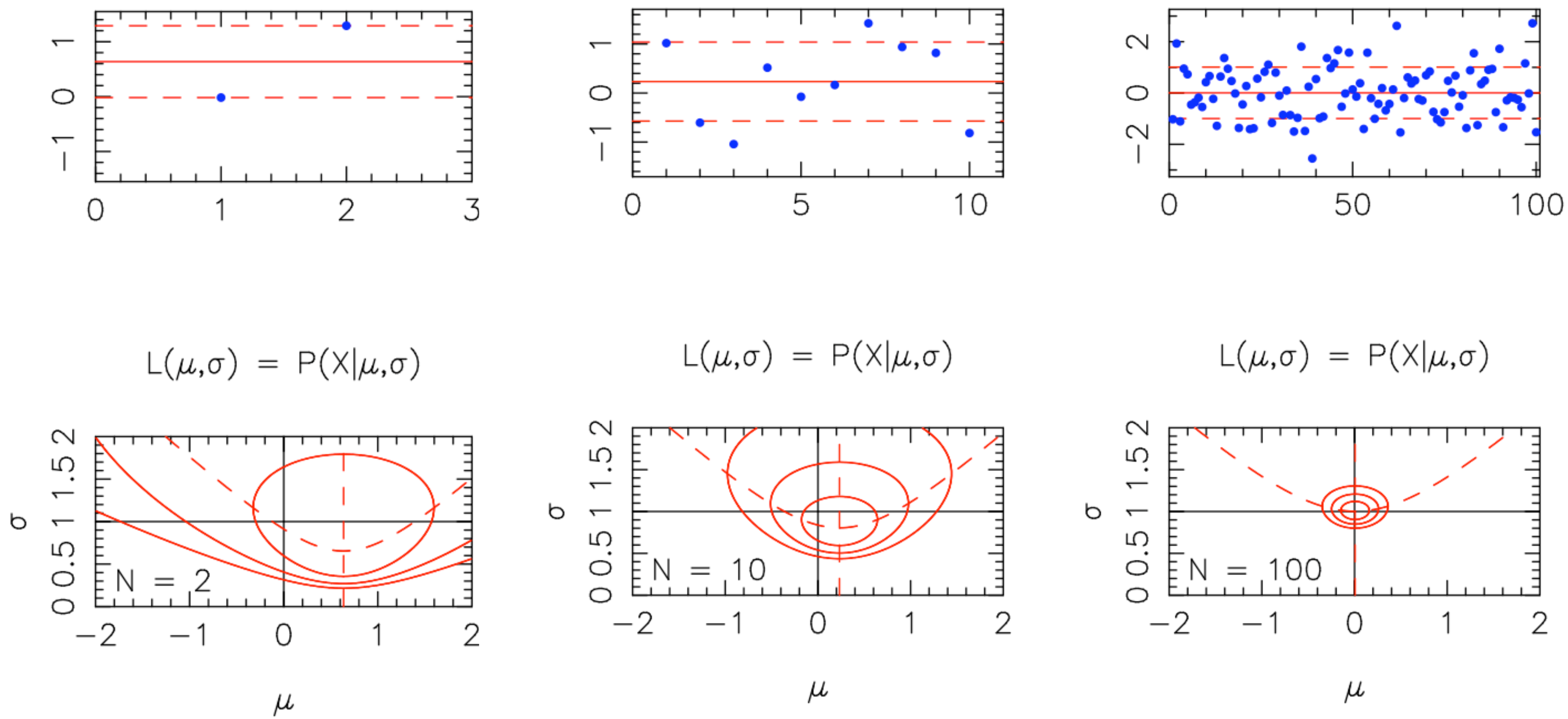


L(μ,σ) = P(X|μ,σ)



Note: ML gives biased estimate for σ.

Example: Estimate both μ and σ

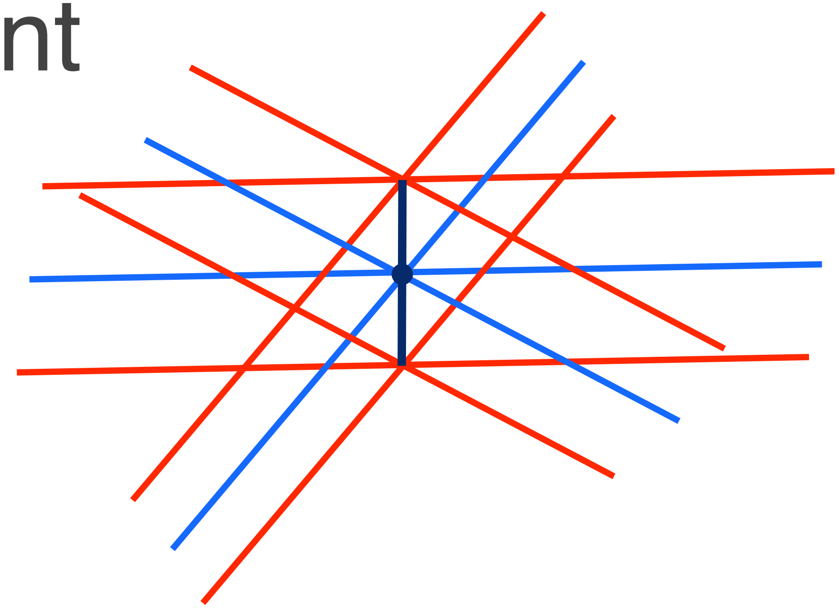


Contours: 1,2,3-sigma 2-parameter confidence regions for μ and σ.

Dashed curves: maximum-likelihood estimates for μ_{ML} and σ_{ML}.

True values: $\mu = 0$ and $\sigma = 1$.

Fit a line to *N=1* data point

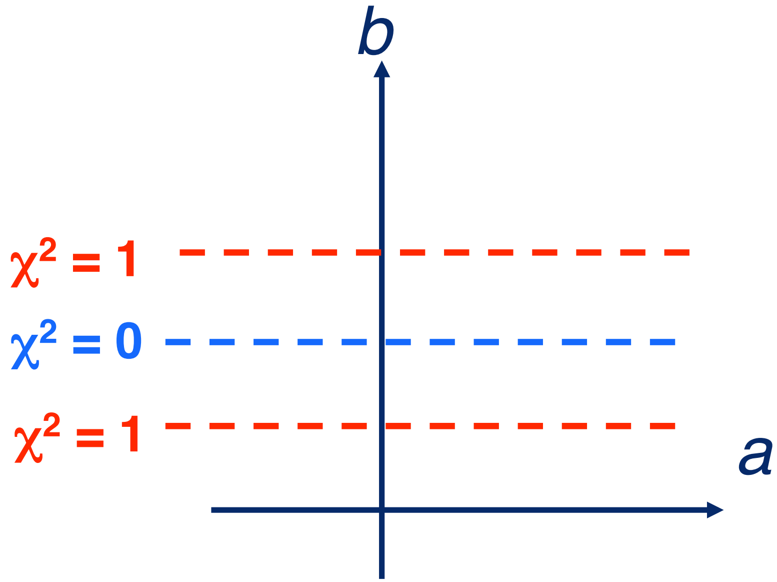


Fit $y = a\,x + b$ to *N=1* data point:

Blue lines : $\chi^2 = 0$

Red lines : $\chi^2 = 1$

χ^2 contours in the *(a,b)* plane:

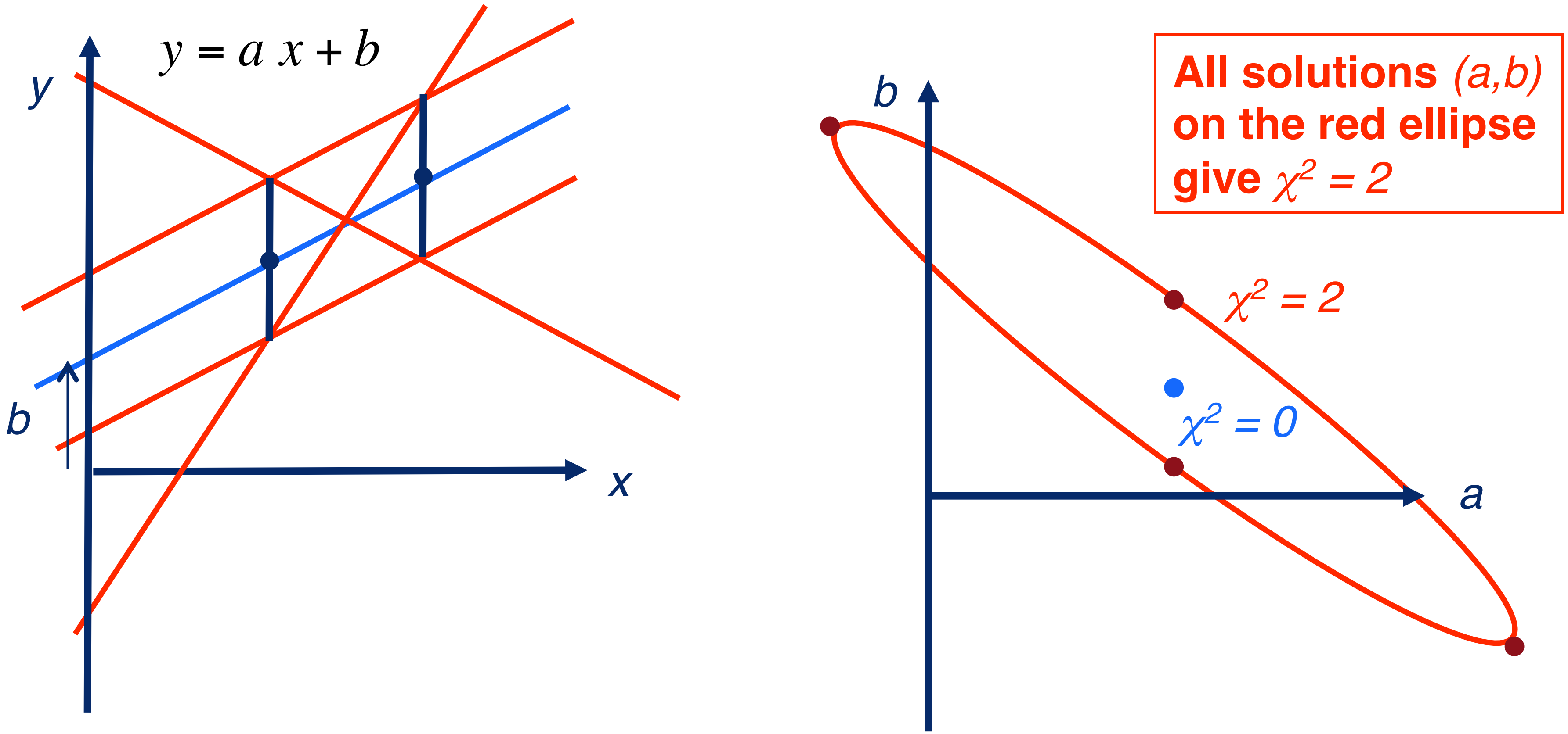


Solution is degenerate, since *M=2* parameters are constrained by only *N=1* data point.

Bayes: prior *P(a,b)* needed to determine a unique solution.

Fit a line to $N = 2$ data points

- Fit $y = a\,x + b$ to $N = 2$ data points:
  - red lines give $\chi^2 = 2$
  - blue line gives $\chi^2 = 0$
- Note that *a, b* are not independent.

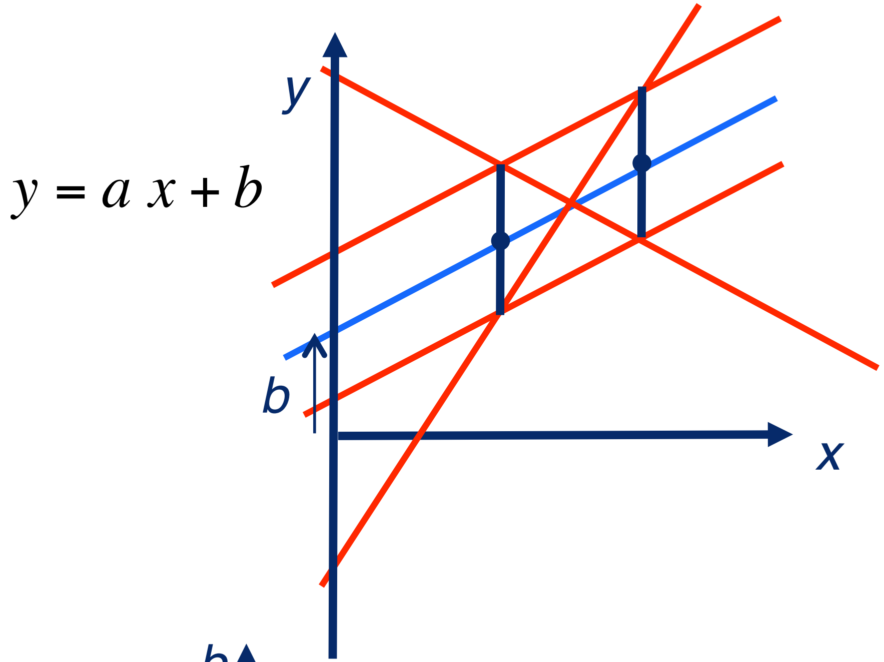


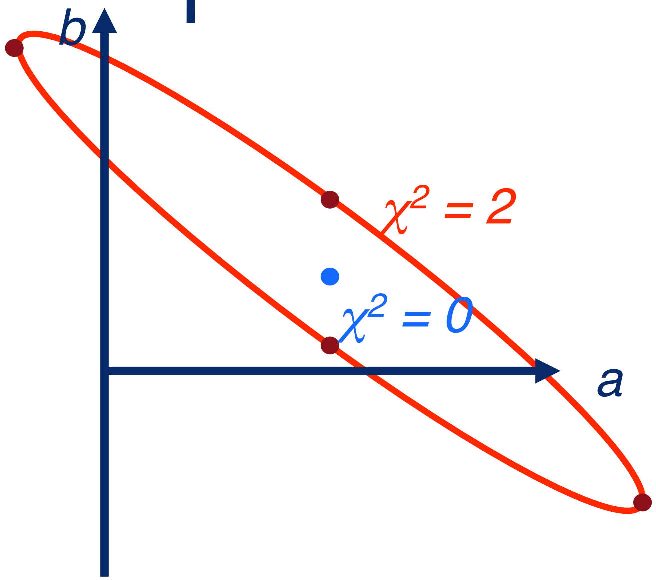
Correlated Parameters

- *a* and *b* are correlated : (
- To find the optimal (a,b) we must:
  - minimize χ^2 with respect to *a* at a sequence of fixed *b*
  - then minimise the resulting χ^2 values with respect to *b*.

- If *a* and *b* were independent, then all slices through the χ^2 surface at each fixed *b* would have same shape and minimum.
- Similarly for *a*.
- We could then optimize *a* and *b* independently, saving a lot of calculation

- How to make *a* and *b* independent of each other?





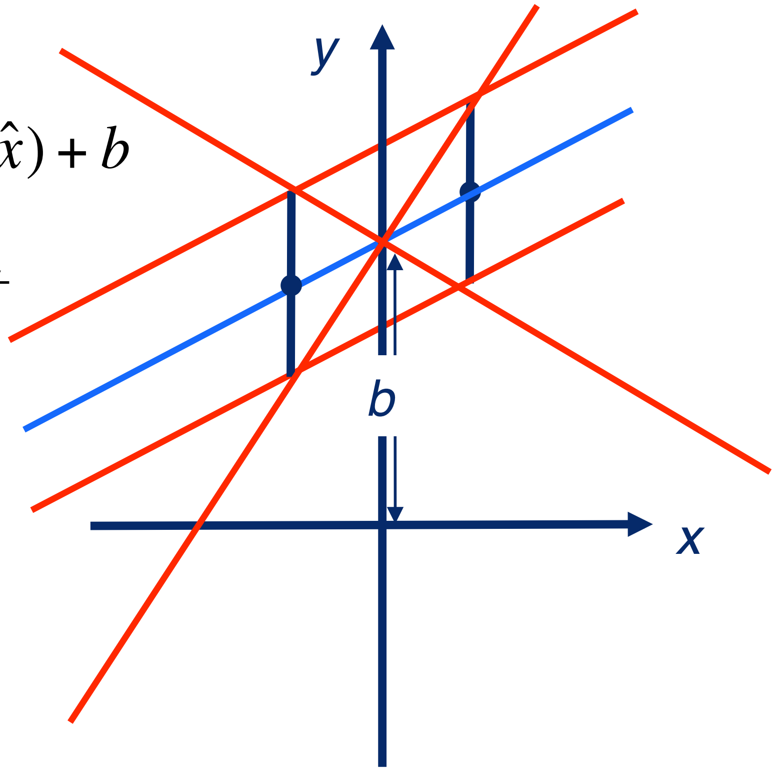
Orthogonal Parameters
for fitting a line to *N = 2* data points

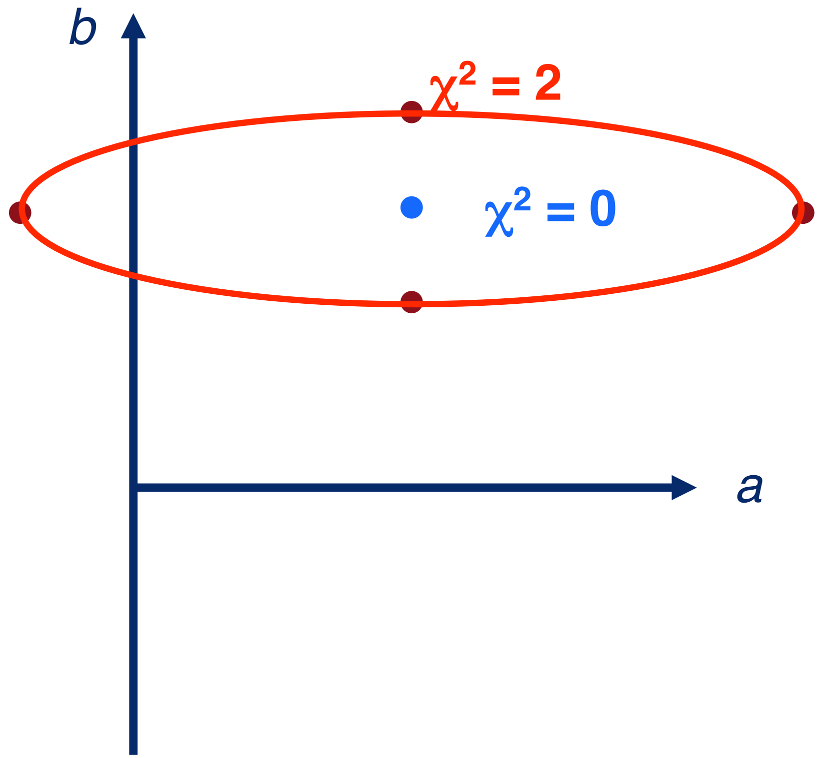
- Fit $y = a(x - \hat{x}) + b$

- Note: *a, b* are now independent !

$$y = a\,(x - \hat{x}) + b$$

$$\hat{x} = \frac{x_1 + x_2}{2}$$

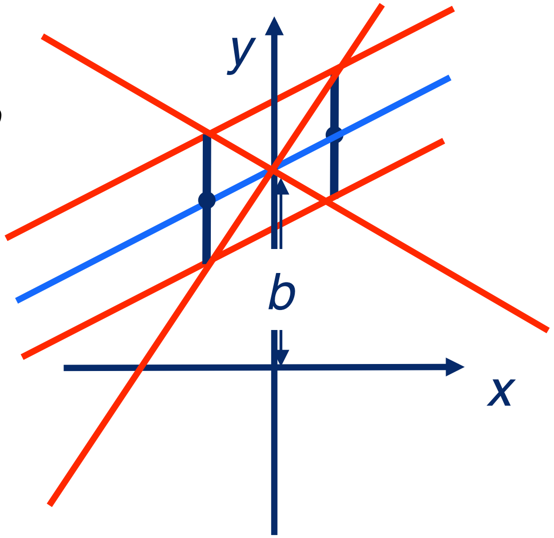




Orthogonal slope and intercept

$$y = a\,(x - \hat{x}) + b$$

$$\hat{x} = \frac{x_1 + x_2}{2}$$

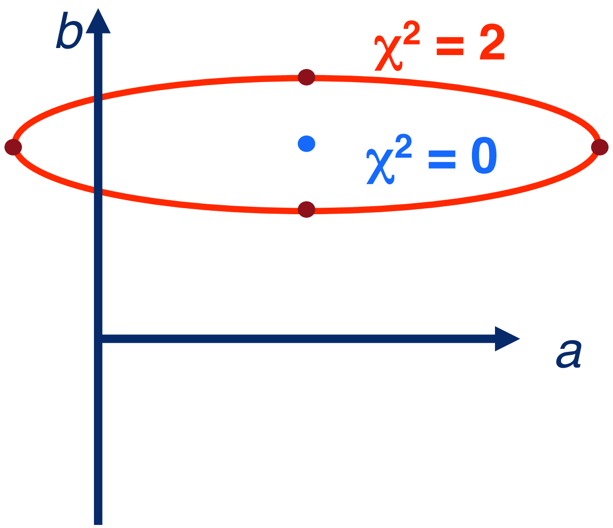


Analysis using the algebra of random variables:

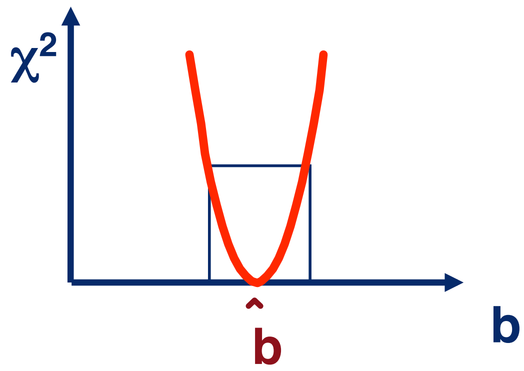
$$\hat{b} = \hat{y} = \frac{y_1 + y_2}{2} \qquad \hat{a} = \frac{y_2 - y_1}{(x_2 - x_1)}$$

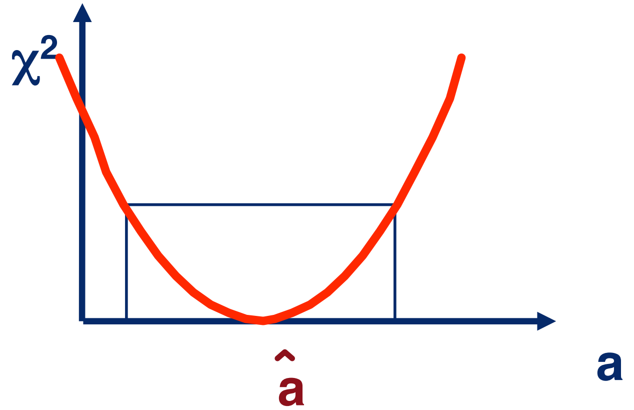
$$\sigma^2(\hat{y}) = \frac{2\sigma^2}{4} \qquad \sigma^2(\hat{a}) = \frac{2\,\sigma^2}{(x_2 - x_1)^2}$$

$$\sigma(\hat{y}) = \frac{\sigma}{\sqrt{2}} \qquad \sigma(\hat{a}) = \sqrt{2}\,\frac{\sigma}{(x_2 - x_1)}$$



Corresponds to $\Delta\chi^2 = 1$.

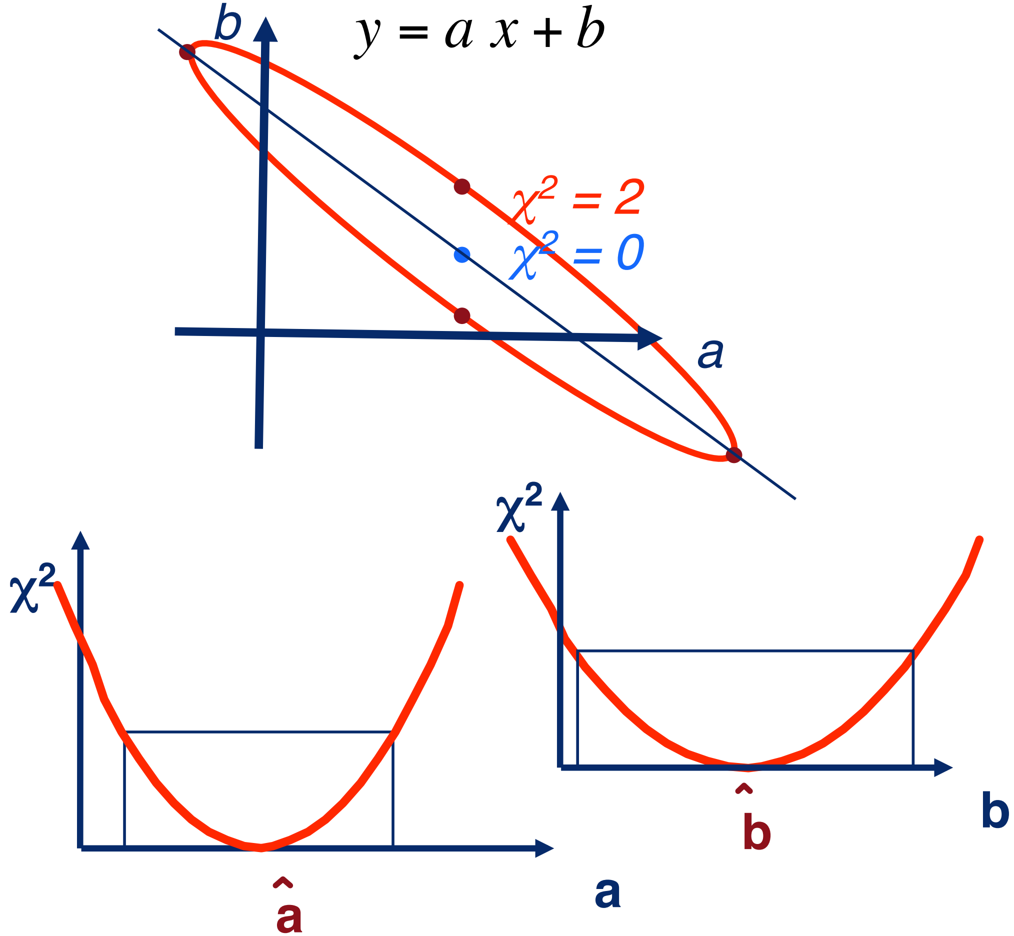




Orthogonal vs Correlated Parameters

Correlated Parameters:

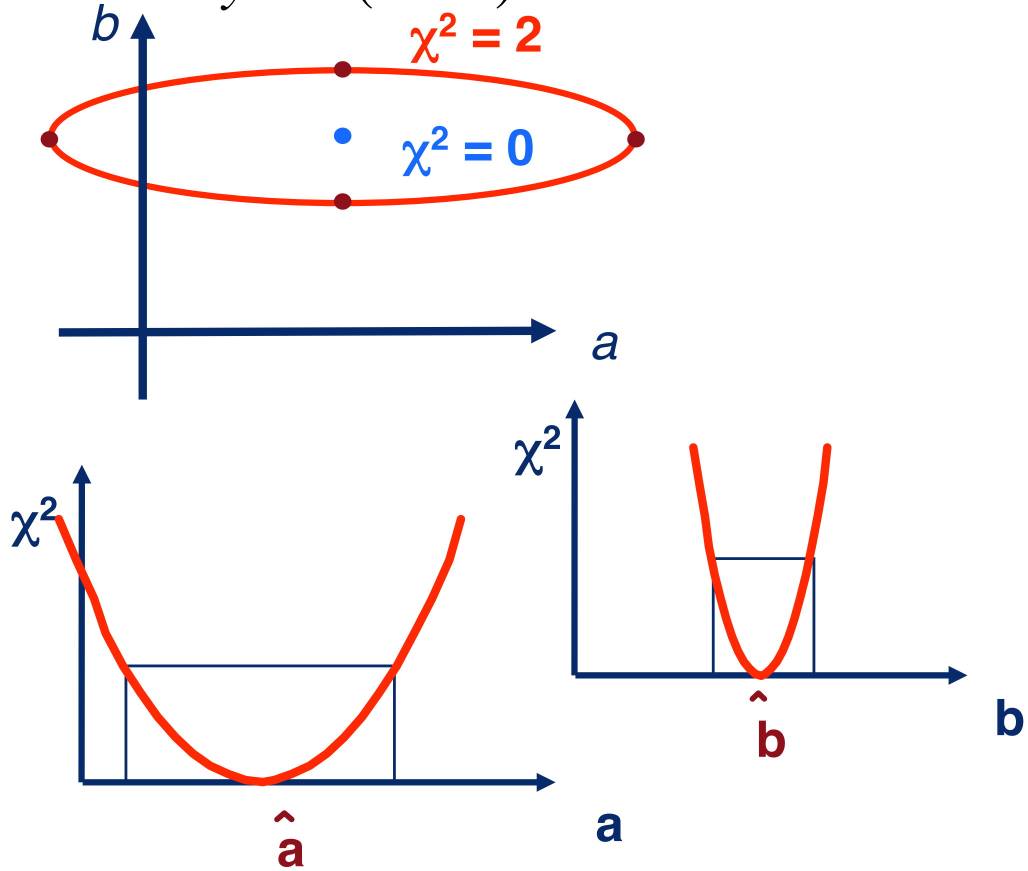
$$y = a\,x + b$$



For each *a*, a different *b* minimises χ^2.
For each *b*, a different *a* minimises χ^2.

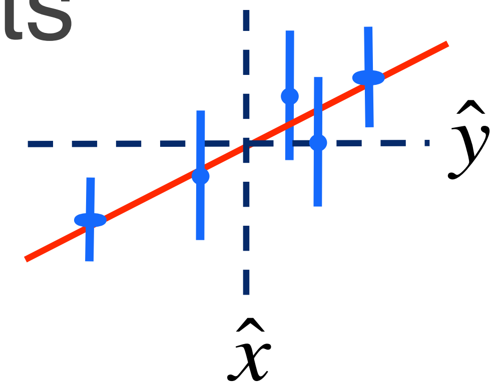
Orthogonal Parameters:

$$y = a\,(x - \hat{x}) + b$$



For any *a*, the same *b* minimises χ^2.
For any *b*, the same *a* minimises χ^2.

Fit a line to *N* data points



- If we use $y = a\,x + b$
  then *a, b* are correlated.
- To make *a, b* orthogonal:

$$y = a\,(x - \hat{x}) + b \qquad \hat{x} = \frac{\sum x_i/\sigma_i^{\,2}}{\sum 1/\sigma_i^{\,2}}$$

- Intercept: **optimal average**:

$$\hat{b} = \hat{y} = \frac{\sum y_i/\sigma_i^{\,2}}{\sum 1/\sigma_i^{\,2}}, \qquad \mathrm{Var}\left[\hat{b}\right] = \frac{1}{\sum 1/\sigma_i^{\,2}}$$

- Slope: **optimal scaling**:

$$\hat{a} = \frac{\sum (y_i - \hat{y})(x_i - \hat{x})/\sigma_i^{\,2}}{\sum (x_i - \hat{x})^2/\sigma_i^{\,2}}, \quad \mathrm{Var}[\hat{a}] = \frac{1}{\sum (x_i - \hat{x})^2/\sigma_i^{\,2}}$$

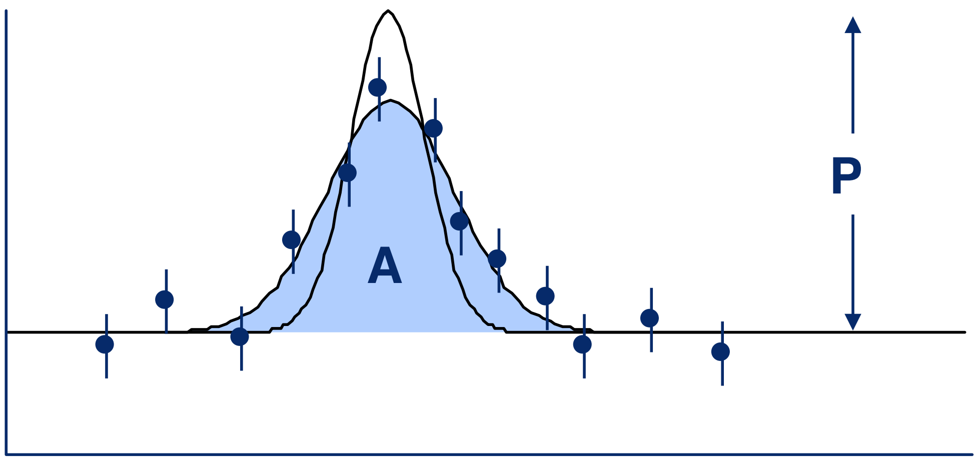
Choose Orthogonal Parameters

- Good practice (when possible).
- Results for any one parameter don't depend on values of other parameters.
- Example: fit a gaussian profile. 2 fit parameters:
  - Width, w
  - Area or peak value. Which is best?

Peak value depends on width – bad

$$f(x) = Pe^{-\frac{1}{2}\left(\frac{x-x_0}{w}\right)^2}$$

$$g(x) = \frac{A}{w\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-x_0}{w}\right)^2}$$



Area is (more nearly) independent of width – good