Variance and Co-variance

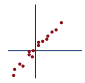
- Variance of $X+Y$ depends on how X and Y co-vary:

$$\boxed{\begin{aligned}\mathrm{Var}(X+Y) &= \mathrm{Var}(X)+\mathrm{Var}(Y)+2\,\mathrm{Cov}(X,Y)\\ \mathrm{Cov}(X,Y) &\equiv \left\langle (X-\langle X\rangle)(Y-\langle Y\rangle)\right\rangle\end{aligned}}$$

$$\begin{aligned}\mathrm{Var}(X+Y) &\equiv \left\langle [X+Y-\langle X+Y\rangle]^2\right\rangle\\ &= \left\langle [X+Y-\langle X\rangle-\langle Y\rangle]^2\right\rangle\\ &= \left\langle [(X-\langle X\rangle)+(Y-\langle Y\rangle)]^2\right\rangle\\ &= \left\langle (X-\langle X\rangle)^2+(Y-\langle Y\rangle)^2+2\,(X-\langle X\rangle)(Y-\langle Y\rangle)\right\rangle\\ &= \left\langle (X-\langle X\rangle)^2\right\rangle+\left\langle (Y-\langle Y\rangle)^2\right\rangle+2\left\langle (X-\langle X\rangle)(Y-\langle Y\rangle)\right\rangle\\ &= \quad \mathrm{Var}(X) \quad + \quad \mathrm{Var}(Y) \quad + \quad 2\,\mathrm{Cov}(X,Y)\end{aligned}$$

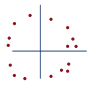
Co-variance vs Independence

- **Cov > 0**

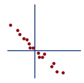


- **Cov = ?**

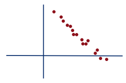
Independent?



- **Cov < 0**

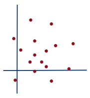


- **Cov = ?**

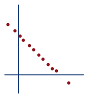


Practice !

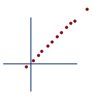
$X = 1 \pm 1$
$Y = 1 \pm 1$
$X + Y = ? \pm ?$



$X = 1 \pm 1$
$Y = 2 - X$
$X + Y = ? \pm ?$



$X = 1 \pm 1$
$Y = X$
$X + Y = ? \pm ?$



$X = 1 \pm 1 \qquad Y = 2 \pm 1 \qquad \mathrm{Cov}[X,Y] = 0 \qquad a = 2 \qquad b = 1$

$Z = aX + bY \qquad \langle Z\rangle = ? \qquad \mathrm{Var}(Z) = ?$

Linear Transformations

- Scale and add any number of random variables:

$$\left\langle \sum_i a_i X_i \right\rangle = \sum_i a_i \langle X_i\rangle \qquad \mathrm{Var}\left[\sum_i a_i X_i\right] = \sum_{i,j} a_i\, a_j \mathrm{Cov}(X_i, X_j)$$

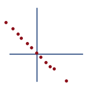
Or, in terms of the (symmetric) **Co-variance Matrix**:

$$\mathrm{Var}\left[\sum_i a_i X_i\right] = \sum_{i,j} a_i C_{ij}\, a_j$$

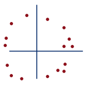
$$\mathrm{Var}\left[(a_i \;\; \ldots \;\; a_N)\begin{pmatrix} X_1 \\ \ldots \\ X_N\end{pmatrix}\right] = (a_i \;\; \ldots \;\; a_N)\begin{pmatrix} C_{11} & \ldots & C_{1N} \\ \ldots & \ldots & \ldots \\ C_{N1} & \ldots & C_{NN}\end{pmatrix}\begin{pmatrix} a_1 \\ \ldots \\ a_N\end{pmatrix}$$

Correlation Coefficient *R(X, Y)*

- **Correlation coefficient**: $\boxed{R(X,Y) \equiv \dfrac{\mathrm{Cov}(X,Y)}{\sigma(X)\,\sigma(Y)}}$



R = -1



R = 0

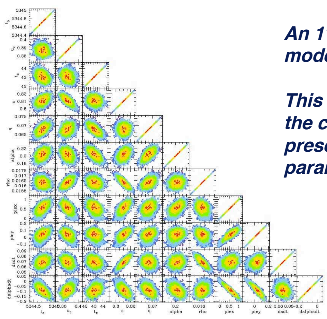
R = +1

- **Correlation matrix**:

$$R_{ij} \equiv \frac{\mathrm{Cov}(X_i, X_j)}{\sigma(X_i)\,\sigma(X_j)} = \begin{bmatrix} 1 & . & . & . \\ . & 1 & . & . \\ . & . & 1 & . \\ . & . & . & 1 \end{bmatrix}$$

- **Variance**: $\mathrm{Var}\left[\sum_i a_i X_i\right] = \sum_i \sum_j a_i a_j \sigma(X_i)\sigma(X_j) R_{ij}$

Example: Correlation Matrix



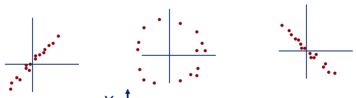
An 11-parameter model fitted to data.

This matrix shows the correlations present among the 11 parameters.

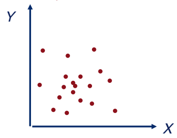
Review: Correlation vs Independence

- **R > 0**
- **R = 0**
- ***R < 0***

Dependent :

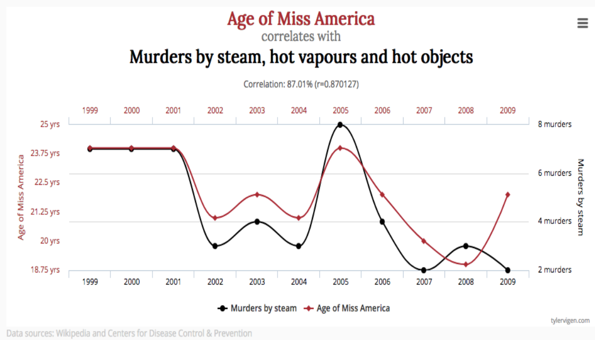


Independent :



$$f(X,Y) = f(X)\, f(Y)$$

Example of a Spurious Correlation



1: Beware Spurious Correlations

- Two variables may appear to be strongly correlated.
- But, can be spurious if you look at many variables, to find the strongest correlations, then pretend you only looked at those.

2 : Correlation is not Causation

- Correlation of 2 variables does not mean that one causes the other. Both could be side effects of something else.

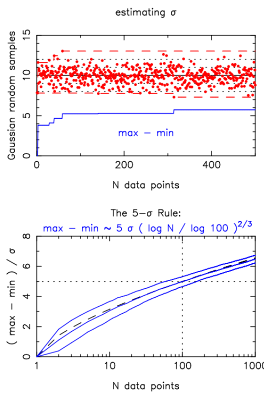
Misleading Significance Claims

If we look at 100 points, we typically find 2 that are 5-sigma apart.

If we pull out those 2

(and omit the others)

we can't honestly claim to have a 5-sigma result.



Review: Algebra of Random Variables

$$\langle a \rangle = a \qquad \mathrm{Var}[a] = 0$$

$$\langle aX \rangle = a \langle X \rangle \qquad \mathrm{Var}[aX] = a^2\, \mathrm{Var}[X]$$

$$\langle X+Y \rangle = \langle X \rangle + \langle Y \rangle \qquad \mathrm{Var}[X+Y] = \mathrm{Var}[X] + \mathrm{Var}[Y] + 2\,\mathrm{Cov}[X,Y]$$

Co - variance :

$$\mathrm{Cov}[X,Y] \equiv \left\langle \left(X - \langle X \rangle \right)\left(Y - \langle Y \rangle \right) \right\rangle \qquad \mathrm{Var}[X] \equiv \mathrm{Cov}[X,X]$$

Linear transformations :

$$\left\langle \sum_i a_i X_i \right\rangle = \sum_i a_i \langle X_i \rangle \qquad \mathrm{Var}\left[\sum_i a_i X_i \right] = \sum_i \sum_j a_i\, a_j\, \sigma_i\, \sigma_j\, R_{ij}$$

Correlation Matrix :

$$R_{ij} \equiv \frac{\mathrm{Cov}(X_i, X_j)}{\sigma_i\, \sigma_j} \qquad \sigma_i \equiv \sigma(X_i)$$

Practice the “fuzzy” algebra of random variables

$$6\ (1 \pm 1) \quad =$$

$$(1 \pm 1) \quad + \quad (2 \pm 2) \quad =$$

$$(1 \pm 2) \quad - \quad (2 \pm 2) \quad =$$

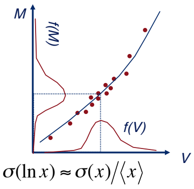
Practice until this becomes automatic …

Functions of Random Variables

Often what we can measure is not what we are most interested in!
Example: mass of binary-star system:

$$M = \frac{V^2 a}{G} = \frac{V^3 P}{2\pi G}$$

We want M, but can only measure V and P.
P = accurate, but V usually less certain.
What is the uncertainty in M ?



For power-laws: $\ln M = 3 \ln V + \ln P + \text{const.} \quad \sigma(\ln x) \approx \sigma(x) / \langle x \rangle$

$$\left(\frac{\sigma_M}{\langle M \rangle}\right)^2 \approx \left(3 \frac{\sigma_V}{\langle V \rangle}\right)^2 + \left(\frac{\sigma_P}{\langle P \rangle}\right)^2$$

(valid for ***small*** and ***independent*** errors in V and P).
How do error bars propagate through non-linear functions?

Functions of a Random Variable

$Y = y(X) \qquad \frac{dY}{dX} = y'(X)$

Conserve probability:

$d(\text{Prob}) = f(Y)\,|dY| = f(X)\,|dX|$

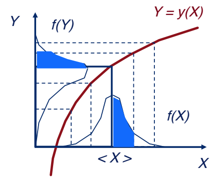
$f(Y) = f(X) \left|\frac{dX}{dY}\right| = \frac{f(X)}{|y'(X)|}$

mean value (biased)

$\langle Y \rangle = y(\langle X \rangle) + \frac{1}{2} y''(\langle X \rangle)\,\sigma_X^2 + \ldots$

standard deviation (stretched)

$\sigma_Y = \sigma_X \left|\frac{dy}{dx}\right|_{X=\langle X \rangle} + \ldots$



Negative curvature:
Long tail for $Y < y(<X>)$
Bias: $<Y> \; < \; y(<X>)$.

Median is not biased:
Med(Y) = y(Med(X))

Examples of Non-linear Transformations

Spectral Energy Distributions: per unit **wavelength** (erg cm^{-2}s^{-1} A^{-1}),
or per unit **frequency** (erg cm^{-2} s^{-1} Hz^{-1})

$f_\nu(\lambda)\,|d\nu| = f_\lambda(\lambda)\,|d\lambda|$

$$\nu = \frac{c}{\lambda} \qquad d\nu = -\frac{c}{\lambda^2} d\lambda \quad \Rightarrow f_\nu(\lambda) = \left|\frac{d\lambda}{d\nu}\right| f_\lambda(\lambda) = \frac{\lambda^2}{c} f_\lambda(\lambda)$$

Converting a **flux** to a **magnitude**:

- Measure Flux: Gaussian distribution: $f \sim G(<f>, \sigma_f^2)$
- Nonlinear transformation induces a bias:

$$m = m_0 - 2.5 \log f$$
$$\langle m \rangle = m_0 - 2.5 \log \langle f \rangle + a\,\sigma_m^2$$

- PROBLEM: evaluate a, σ_m in terms of $<f>$, σ_f.