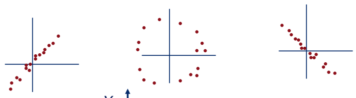
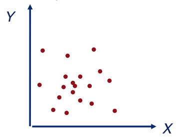
Review: Correlation vs Independence

- **R > 0**
- **R = 0**
- ***R < 0***

Dependent :



Independent :



$$f(X,Y) = f(X)\,f(Y)$$

Review: Algebra of Random Variables

$$\langle a \rangle = a \qquad \mathrm{Var}[a] = 0$$

$$\langle a\,X \rangle = a\langle X \rangle \qquad \mathrm{Var}[a\,X] = a^2\,\mathrm{Var}[X]$$

$$\langle X+Y \rangle = \langle X \rangle + \langle Y \rangle \qquad \mathrm{Var}[X+Y] = \mathrm{Var}[X] + \mathrm{Var}[Y] + 2\,\mathrm{Cov}[X,Y]$$

Co - variance :

$$\mathrm{Cov}[X,Y] \equiv \Big\langle \big(X - \langle X \rangle\big)\big(Y - \langle Y \rangle\big) \Big\rangle \qquad \mathrm{Var}[X] = \mathrm{Cov}[X,X]$$

Linear transformations :

$$\left\langle \sum_i a_i X_i \right\rangle = \sum_i a_i \langle X_i \rangle \qquad \mathrm{Var}\left[\sum_i a_i X_i\right] = \sum_i \sum_j a_i\,a_j\,\sigma_i\,\sigma_j\,R_{ij}$$

Correlation Matrix :

$$R_{ij} \equiv \frac{\mathrm{Cov}(X_i, X_j)}{\sigma_i\,\sigma_j} \qquad \sigma_i \equiv \sigma(X_i)$$

Practice the “fuzzy” algebra of random variables

$$6\;(1 \pm 1) \quad =$$

$$(1 \pm 1) \quad + \quad (2 \pm 2) \quad =$$

$$(1 \pm 2) \quad - \quad (2 \pm 2) \quad =$$

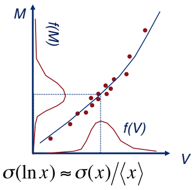
Practice until this becomes automatic …

Functions of Random Variables

Often what we can measure is not what we are most interested in!
Example: mass of binary-star system:

$$M = \frac{V^2 a}{G} = \frac{V^3 P}{2\pi G}$$

We want *M*, but can only measure *V* and *P*.
P = accurate, but *V* usually less certain.
What is the uncertainty in *M* ?



For power-laws: $\ln M = 3\ln V + \ln P + \mathrm{const.} \qquad \sigma(\ln x) \approx \sigma(x)/\langle x \rangle$

$$\left(\frac{\sigma_M}{\langle M \rangle}\right)^2 \approx \left(3\frac{\sigma_V}{\langle V \rangle}\right)^2 + \left(\frac{\sigma_P}{\langle P \rangle}\right)^2$$

(valid for ***small*** and ***independent*** errors in *V* and *P*).
How do error bars propagate through non-linear functions?

Functions of a Random Variable

$$Y = y(X) \qquad \frac{dY}{dX} = y'(X)$$

Conserve probability:

$$d(\mathrm{Prob}) = f(Y)\,|\mathrm{d}Y| = f(X)\,|\mathrm{d}X|$$

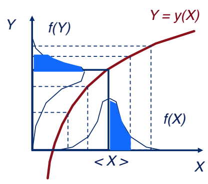
$$f(Y) = f(X)\left|\frac{dX}{dY}\right| = \frac{f(X)}{|y'(X)|}$$

mean value (biased)

$$\langle Y \rangle = y\big(\langle X \rangle\big) + \frac{1}{2}\,y''\big(\langle X \rangle\big)\,\sigma_X^2 + \quad \ldots$$

standard deviation (stretched)

$$\sigma_Y = \sigma_X \left|\frac{dy}{dx}\right|_{X=\langle X \rangle} + \quad \ldots$$



Negative curvature:

Long tail for *Y* < *y*(< *X* >)

Bias: <Y> < y(< X >).

Median is not biased:
Med*(Y) = y(* Med*(X))*

Examples of Non-linear Transformations

Spectral Energy Distributions: per unit **wavelength** (erg $\mathrm{cm^{-2} s^{-1}\,A^{-1}}$),
or per unit **frequency** (erg $\mathrm{cm^{-2}\,s^{-1}\,Hz^{-1}}$)

$$f_\nu(\lambda)\,|\mathrm{d}\nu| = f_\lambda(\lambda)\,|\mathrm{d}\lambda|$$

$$\nu = \frac{c}{\lambda} \qquad \mathrm{d}\nu = -\frac{c}{\lambda^2}\mathrm{d}\lambda \quad \Rightarrow f_\nu(\lambda) = \left|\frac{\mathrm{d}\lambda}{\mathrm{d}\nu}\right| f_\lambda(\lambda) = \frac{\lambda^2}{c}\,f_\lambda(\lambda)$$

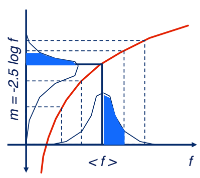
Converting a **flux** to a **magnitude**:

- Measure Flux: Gaussian distribution: $f \sim G(\langle f \rangle, \sigma_f^2)$
- Nonlinear transformation induces a bias:

$$m = m_0 - 2.5\,\log f$$

$$\langle m \rangle = m_0 - 2.5\,\log\langle f \rangle + a\,\sigma_m^2$$

- PROBLEM: evaluate a, σ_m in terms of $\langle f \rangle$, σ_f.

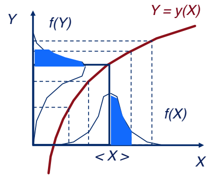


Nonlinear Transformations:
A Bias from Curvature + Noise

Taylor expand $Y = y(X)$ around $X = \langle X \rangle$:

$$y(X) = y(\langle X \rangle) + y'(\langle X \rangle)\,\varepsilon + \frac{1}{2} y''(\langle X \rangle)\,\varepsilon^2 + \ldots$$

$$\text{where} \quad \varepsilon \equiv X - \langle X \rangle, \quad \langle \varepsilon \rangle = 0, \quad \langle \varepsilon^2 \rangle = \sigma_X^{\,2}.$$



Hence

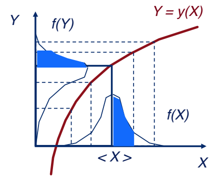
$$\langle y(X) \rangle = \left\langle y(\langle X \rangle) + y'(\langle X \rangle)\,\varepsilon + \frac{1}{2} y''(\langle X \rangle)\,\varepsilon^2 + \ldots \right\rangle$$

$$= y(\langle X \rangle) + y'(\langle X \rangle)\,\langle \varepsilon \rangle + \frac{1}{2} y''(\langle X \rangle)\,\langle \varepsilon^2 \rangle + \ldots$$

$$= y(\langle X \rangle) + \quad 0 \quad \boxed{+\frac{1}{2} y''(\langle X \rangle)\;\sigma_X^{\,2}} + \ldots$$

This is the bias.

Variance of a Transformed Variable



Tangent-curve approximation :
σ(y(x)) = σ(x) **stretched** by a factor *|dy/dx|.*

$$\sigma^2(Y) \equiv \left\langle (Y - \langle Y \rangle)^2 \right\rangle = \left\langle \left[y(\langle X \rangle) + y'(\langle X \rangle)\,\varepsilon + \frac{1}{2} y''(\langle X \rangle)\,\varepsilon^2 + \ldots \right.\right.$$

$$\left.\left. - y(\langle X \rangle) \quad - \quad 0 \quad - \frac{1}{2} y''(\langle X \rangle)\,\sigma^2(X) - \ldots \right]^2 \right\rangle$$

$$= \left\langle \left[y'(\langle X \rangle)\,\varepsilon + O(\varepsilon^2) \right]^2 \right\rangle = \boxed{\left[y'(\langle X \rangle) \right]^2 \sigma_X^2} \quad + \quad \ldots$$

Could extend to higher-order terms (skew, kurtosis) if needed,
but fast computers make it easier to use Monte-Carlo error propagation.

Magnitude Bias

Observe flux: $\quad f \sim G\left(f_0,\ \sigma_f^2 \right) = \left(f_0 \pm \sigma_f \right)$

Transform to magnitude: $\quad m(f) \equiv m_0 - 2.5 \log f$

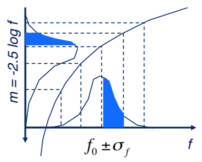
Derivatives: $(\log f = \log e \ln f)$ $\quad \dfrac{\mathrm{d}m}{\mathrm{d}f} = -\dfrac{2.5 \log e}{f}, \quad \dfrac{\mathrm{d}^2 m}{\mathrm{d}f^2} = \dfrac{2.5 \log e}{f^2}.$

$$\sigma_m \approx \left| \frac{\mathrm{d}m}{\mathrm{d}f} \right|_{\langle f \rangle} \sigma_f = \frac{2.5 \log e}{\langle f \rangle} \sigma_f \approx 1.08 \frac{\sigma_f}{\langle f \rangle}.$$

$$\langle m \rangle = m\left(\langle f \rangle \right) + \frac{1}{2} \frac{\mathrm{d}^2 m}{\mathrm{d}f^2}\bigg|_{\langle f \rangle} \sigma_f^2 + \ldots$$

$$= m_0 - 2.5 \log \langle f \rangle + \frac{2.5 \log e}{2 \langle f \rangle^2} \sigma_f^2$$

$$= m_0 - 2.5 \log \langle f \rangle + \frac{\sigma_m^2}{5 \log e}$$



Note the bias toward faint magnitudes.

Magnitude Bias

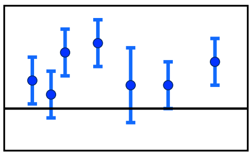
$$m(f) \equiv m_0 - 2.5 \log f$$

$$\sigma_m = (2.5 \log e) \frac{\sigma_f}{\langle f \rangle} \approx 1.08 \frac{\sigma_f}{\langle f \rangle}.$$

$$\langle m \rangle = m\left(\langle f \rangle \right) + \mathrm{bias}\left(\langle m \rangle \right)$$

$$\mathrm{bias}\left(\langle m \rangle \right) = \frac{\sigma_m^2}{5 \log e} \approx 0.01 \left(\frac{\sigma_m}{0.15} \right)^2$$

15% erorrs -> 1% bias
50% errors -> 10% bias



Given noisy fluxes, should you average fluxes and then compute the magnitude?

$$m(\langle f \rangle) = m_0 - 2.5 \log \langle f \rangle$$

Or, should you convert fluxes to magnitudes and average the magnitudes ?

$$\langle m(f) \rangle = \langle m_0 - 2.5 \log f \rangle$$

Distance from Parallax measurements

Parallax is the apparent motion of stars as the Earth orbits the Sun.

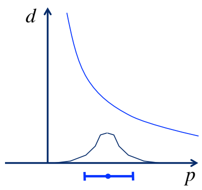
$$\frac{d}{\text{parsec}} = \left(\frac{p}{\text{arcsec}} \right)^{-1}$$

Measure a parallax, with Gaussian error,

$$p = p_0 \pm \sigma_p$$

Estimate the distance and its uncertainty:

$$d = \frac{1}{p_0} + bias \pm \sigma_d$$



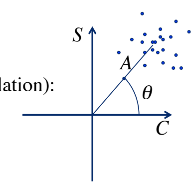
Include a correction for the bias due to the non-linear transformation.

Cartesian -> Polar coordinates
e.g. Amplitude and Phase

Independent measurements of C and S
(e.g. cos and sin amplitudes of an oscillation):

$$S \;=\; A \sin\theta \;\sim\; \left(S_0 \pm \sigma_S \right)$$

$$C \;=\; A \cos\theta \;\sim\; \left(C_0 \pm \sigma_C \right)$$



Transform to amplitude and phase:

$A \;=\; ? \;\pm\; ? \qquad \theta \;=\; ? \;\pm\; ?$

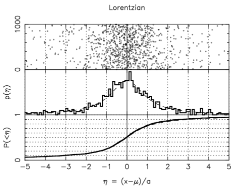
Transforming Random Numbers

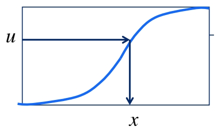
Uniform $\rightarrow$ Lorentzian

$u \sim U(0,1) \rightarrow x \sim L(\mu,\sigma)$

$$u = F(x) = \frac{1}{\pi}\arctan\left[\frac{x-\mu}{\sigma}\right] + \frac{1}{2}$$

$$x = F^{-1}(u) = \mu + \sigma \tan\left[\pi\left(u - \frac{1}{2}\right)\right]$$





Practice :

Uniform $\rightarrow$ Exponential

Uniform $\rightarrow$ Power - law

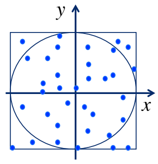
Box-Muller Transform

For Gaussians, cumulative probability *F(x)* has no analytic expression. ☹
Harder to generate Gaussian random numbers $x = F^{-1}(u)$
from Uniform random numbers u.

Two independent uniform random numbers:

$$x \sim U(-1,+1) \qquad y \sim U(-1,+1)$$

Keep if $r^2 = x^2 + y^2 < 1$ and $r > 0$.



Two independent gaussian random numbers:

$$G_1 = \frac{2\,x}{r}\left(-\ln r\right)^{1/2} \qquad G_2 = \frac{2\,y}{r}\left(-\ln r\right)^{1/2}$$

$$r = 0 \rightarrow G = \infty$$

$$r = 1 \rightarrow G = 0$$

G_1 and G_2 have mean 0 and variance 1:

$G_1 \sim G(0,1) \quad G_2 \sim G(0,1)$