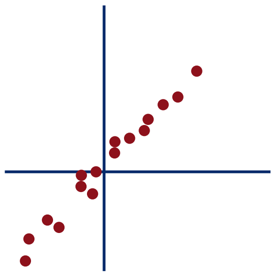
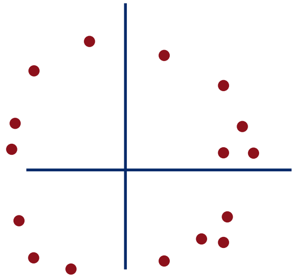
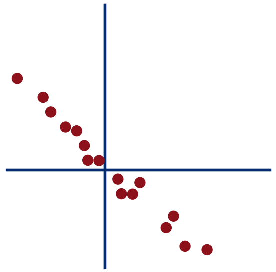
Review: Correlation vs Independence

- **R > 0**
- **R = 0**
- ***R < 0***

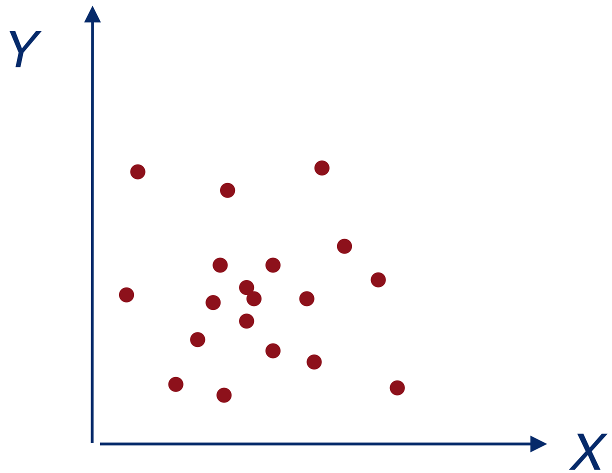
Dependent :





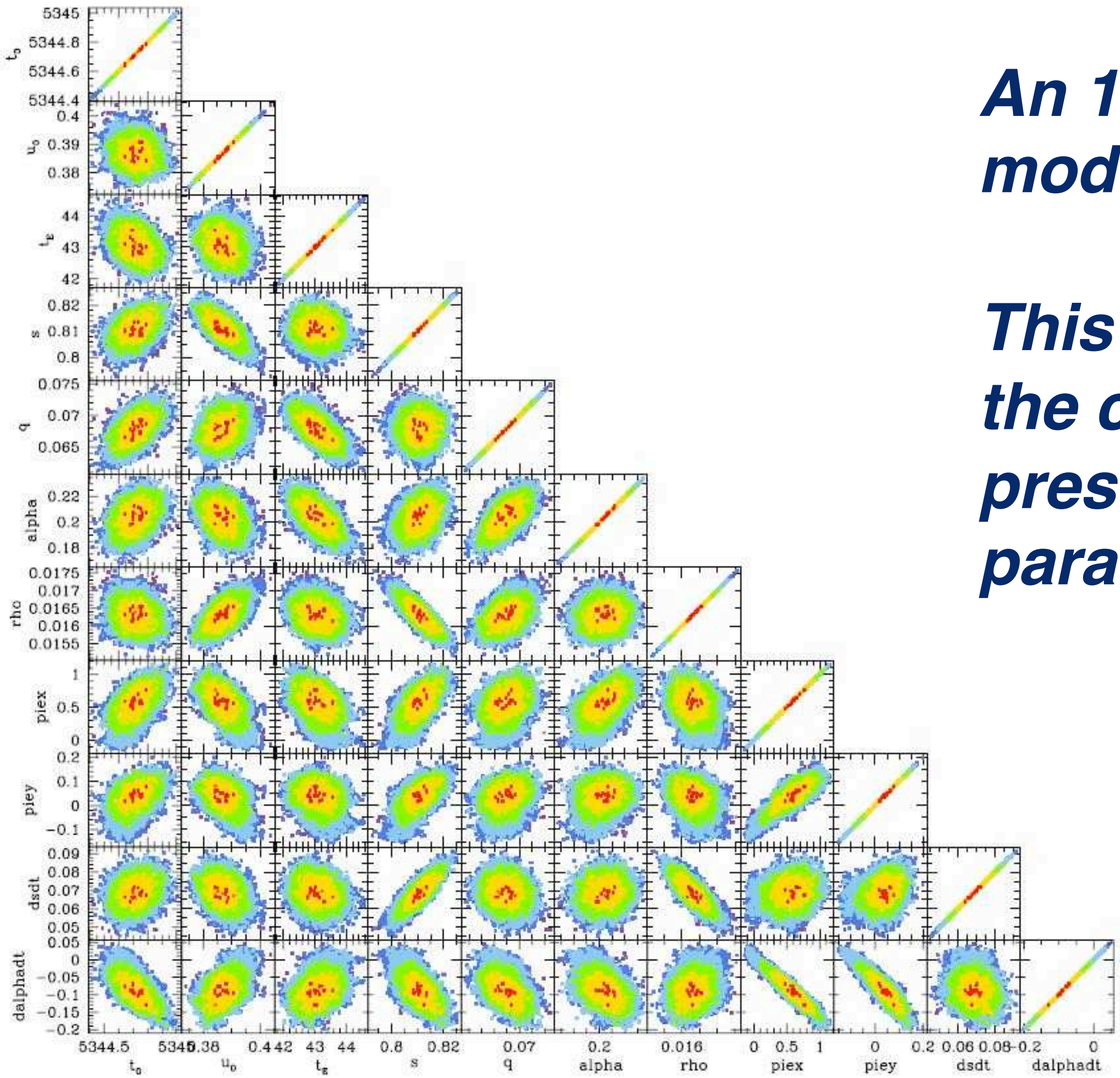


Independent :



$$f(X,Y) = f(X)\, f(Y)$$

Example: Correlation Matrix



An 11-parameter model fitted to data.

This matrix shows the correlations present among the 11 parameters.

Review: Algebra of Random Variables

$$\langle a \rangle = a \qquad \mathrm{Var}[a] = 0$$

$$\langle a X \rangle = a \langle X \rangle \qquad \mathrm{Var}[a X] = a^2 \,\mathrm{Var}[X]$$

$$\langle X + Y \rangle = \langle X \rangle + \langle Y \rangle \qquad \mathrm{Var}[X + Y] = \mathrm{Var}[X] + \mathrm{Var}[Y] + 2\,\mathrm{Cov}[X, Y]$$

Co - variance :

$$\mathrm{Cov}[X, Y] \equiv \left\langle \left(X - \langle X \rangle \right) \left(Y - \langle Y \rangle \right) \right\rangle \qquad \mathrm{Var}[X] \equiv \mathrm{Cov}[X, X]$$

Linear transformations :

$$\left\langle \sum_i a_i X_i \right\rangle = \sum_i a_i \langle X_i \rangle \qquad \mathrm{Var}\left[\sum_i a_i X_i \right] = \sum_i \sum_j a_i \, a_j \, \sigma_i \, \sigma_j \, R_{ij}$$

Correlation Matrix :

$$R_{ij} \equiv \frac{\mathrm{Cov}(X_i, X_j)}{\sigma_\mathrm{i} \, \sigma_\mathrm{j}} \qquad \sigma_\mathrm{i} \equiv \sigma(X_i)$$

Practice the “fuzzy” algebra of random variables

$$6\,(1 \pm 1) \quad =$$

$$(1 \pm 1) \quad + \quad (2 \pm 2) \quad =$$

$$(1 \pm 2) \quad - \quad (2 \pm 2) \quad =$$

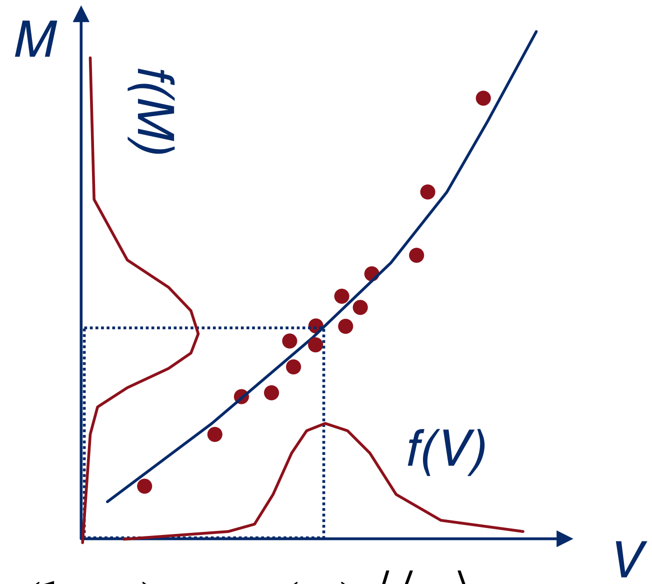
Practice until this becomes automatic …

Functions of Random Variables

Often what we can measure is not what we are most interested in!

Example: mass of binary-star system:

$$M = \frac{V^2 a}{G} = \frac{V^3 P}{2\pi G}$$



We want *M,* but can only measure *V* and *P*.

P = accurate,   but *V*   usually less certain.

What is the uncertainty in *M* ?

For power-laws: $\ln M = 3\ln V + \ln P + \text{const.} \quad \sigma(\ln x) \approx \sigma(x) / \langle x \rangle$

$$\left(\frac{\sigma_M}{\langle M \rangle} \right)^2 \approx \left(3\frac{\sigma_V}{\langle V \rangle} \right)^2 + \left(\frac{\sigma_P}{\langle P \rangle} \right)^2$$

(valid for *small* and *independent* errors in *V* and *P*).

How do error bars propagate through non-linear functions?

Functions of a Random Variable

$$Y = y(X) \qquad \frac{dY}{dX} = y'(X)$$

conserve probability:

$$d(\text{Prob}) = f(Y)\,\left|\text{d}Y\right| = f(X)\,\left|\text{d}X\right|$$

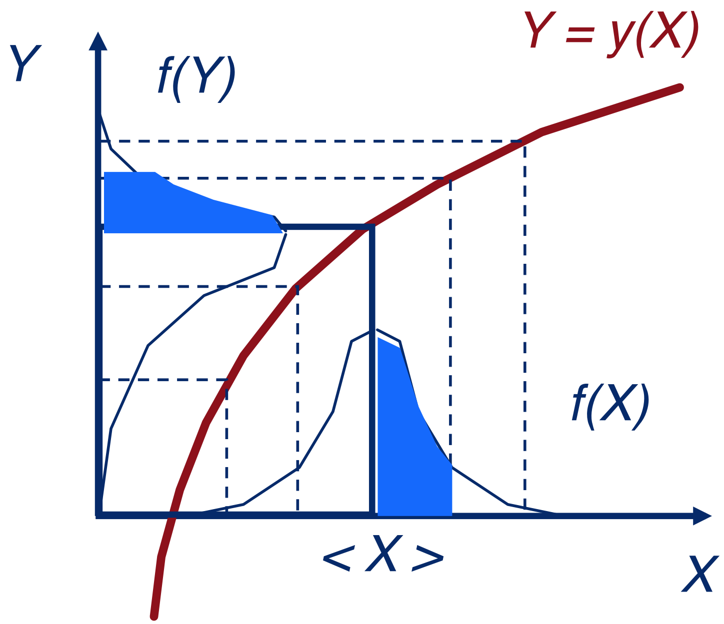
$$f(Y) = f(X)\left|\frac{dX}{dY}\right| = \frac{f(X)}{\left|y'(X)\right|}$$

mean value (biased)

$$\left\langle Y \right\rangle = y\left(\left\langle X \right\rangle\right) + \frac{1}{2}\, y''\left(\left\langle X \right\rangle\right)\sigma_X^2 + \quad \ldots$$

standard deviation (stretched)

$$\sigma_Y = \sigma_X \left|\frac{dy}{dx}\right|_{X=\langle X \rangle} + \quad \ldots$$



Negative curvature:

Long tail for $Y < y(<X>)$

Bias: $<Y> < y(<X>)$.

Median is not biased:
Med(Y) = $y($ Med(X) $)$

Examples of Non-linear Transformations

Spectral Energy Distributions: per unit **wavelength** (ergcm^{-2}A^{-1}),
or per unit **frequency** (erg cm^{-2} s^{-1} Hz^{-1})

$$f_\nu(\lambda)\,|\mathrm{d}\nu| = f_\lambda(\lambda)\,|\mathrm{d}\lambda|$$

$$\nu = \frac{c}{\lambda} \qquad \mathrm{d}\nu = -\frac{c}{\lambda^2}\mathrm{d}\lambda \quad \Rightarrow f_\nu(\lambda) = \left|\frac{\mathrm{d}\lambda}{\mathrm{d}\nu}\right| f_\lambda(\lambda) = \frac{\lambda^2}{c} f_\lambda(\lambda)$$

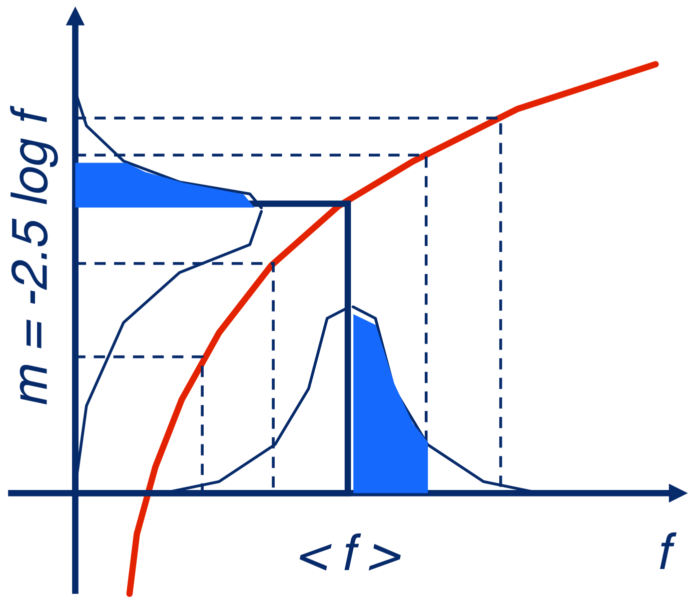
Converting a **flux** to a **magnitude**:

– Measure Flux: Gaussian distribution: $f \sim G(\langle f\rangle, \sigma_f^{\,2})$
– Nonlinear transformation induces a bias:

$$m = m_0 - 2.5 \log f$$

$$\langle m \rangle = m_0 - 2.5 \log \langle f \rangle + a\,\sigma_m^2$$

– PROBLEM: evaluate a, σ_m in terms of $\langle f\rangle$, σ_f.

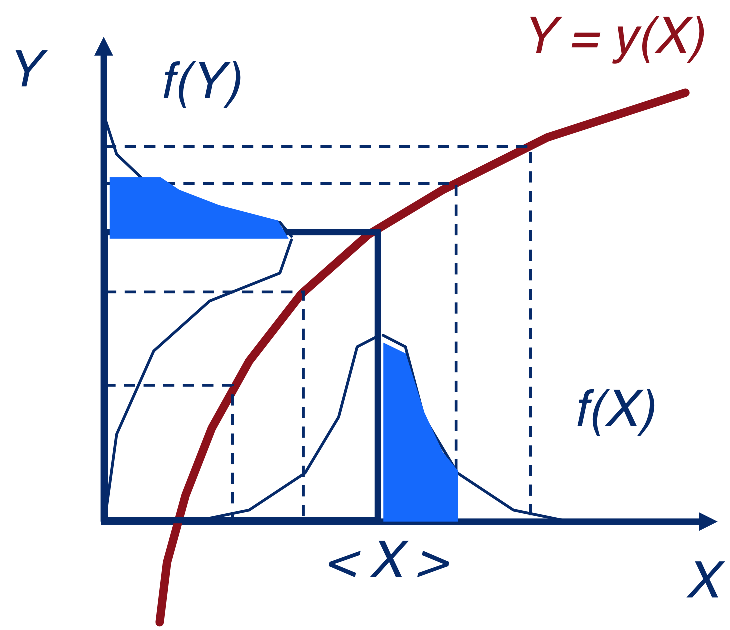


Nonlinear Transformations: A Bias from Curvature + Noise

Taylor expand $Y = y(X)$ around $X = \langle X \rangle$:

$$y(X) = y(\langle X \rangle) + y'(\langle X \rangle)\,\varepsilon + \frac{1}{2}\,y''(\langle X \rangle)\,\varepsilon^2 + \ldots$$

$$\text{where} \quad \varepsilon \equiv X - \langle X \rangle, \quad \langle \varepsilon \rangle = 0, \quad \langle \varepsilon^2 \rangle = \sigma_X{}^2 .$$



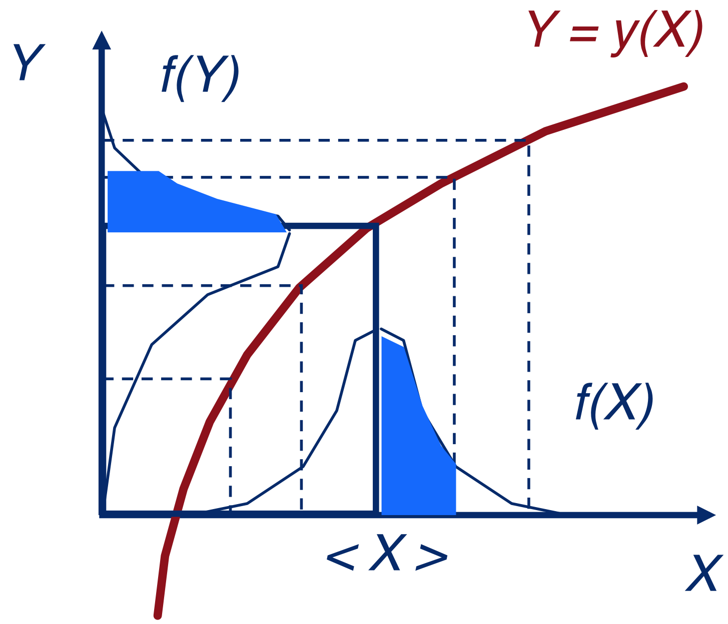
Hence

$$\langle y(X) \rangle = \left\langle y(\langle X \rangle) + y'(\langle X \rangle)\,\varepsilon + \frac{1}{2}\,y''(\langle X \rangle)\,\varepsilon^2 + \ldots \right\rangle$$

$$= y(\langle X \rangle) + y'(\langle X \rangle)\,\langle \varepsilon \rangle + \frac{1}{2}\,y''(\langle X \rangle)\,\langle \varepsilon^2 \rangle + \ldots$$

$$= y(\langle X \rangle) + \quad 0 \quad + \frac{1}{2}\,y''(\langle X \rangle)\,\sigma_X{}^2 + \ldots$$

This is the bias.



Variance of a Transformed Variable

Tangent-curve approximation :

$\sigma(y(x)) = \sigma(x)$ **stretched** by a factor $|dy/dx|$.

$$\sigma^2(Y) \equiv \left\langle (Y - \langle Y \rangle)^2 \right\rangle = \left\langle \left[y(\langle X \rangle) + y'(\langle X \rangle)\,\varepsilon + \frac{1}{2} y''(\langle X \rangle)\,\varepsilon^2 + \ldots \right.\right.$$

$$\left.\left. - \, y(\langle X \rangle) \quad - \quad 0 \quad - \frac{1}{2} y''(\langle X \rangle)\,\sigma^2(X) - \ldots \right]^2 \right\rangle$$

$$= \left\langle \left[\, y'(\langle X \rangle)\,\varepsilon + O(\varepsilon^2) \, \right]^2 \right\rangle = \left[y'(\langle X \rangle) \right]^2 {\sigma_X}^2$$

Can extend to higher-order terms (skew, kurtosis) if needed.

Magnitude Bias

Observe flux: $f \sim G\left(f_0,\ \sigma_f^2 \right) = \left(f_0 \pm \sigma_f \right)$

Transform to magnitude: $m(f) \equiv m_0 - 2.5 \log f$

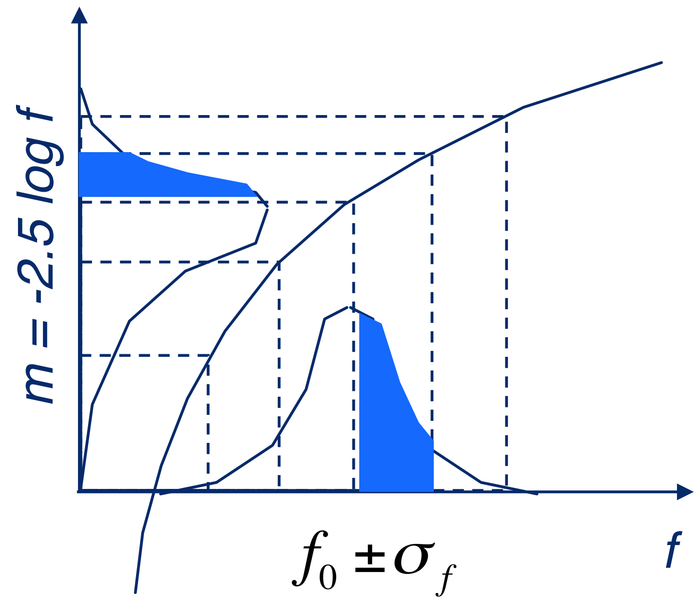
Derivatives: $(\log f = \log e \ln f)$ $\dfrac{\mathrm{d}m}{\mathrm{d}f} = -\dfrac{2.5 \log e}{f}, \quad \dfrac{\mathrm{d}^2 m}{\mathrm{d}f^2} = \dfrac{2.5 \log e}{f^2}.$

$$\sigma_m \approx \left| \frac{\mathrm{d}m}{\mathrm{d}f} \right|_{\langle f \rangle} \sigma_f = \frac{2.5 \log e}{\langle f \rangle} \sigma_f \approx 1.08 \, \frac{\sigma_f}{\langle f \rangle}.$$

$$\langle m \rangle = m\left(\langle f \rangle \right) + \frac{1}{2} \frac{\mathrm{d}^2 m}{\mathrm{d}f^2} \bigg|_{\langle f \rangle} \sigma_f^2$$

$$= m_0 - 2.5 \log \langle f \rangle + \frac{2.5 \log e}{2 \, \langle f \rangle^2} \sigma_f^2$$

$$= m_0 - 2.5 \log \langle f \rangle + \frac{\sigma_m^2}{5 \log e}$$



Note the bias toward faint magnitudes.

Magnitude Bias

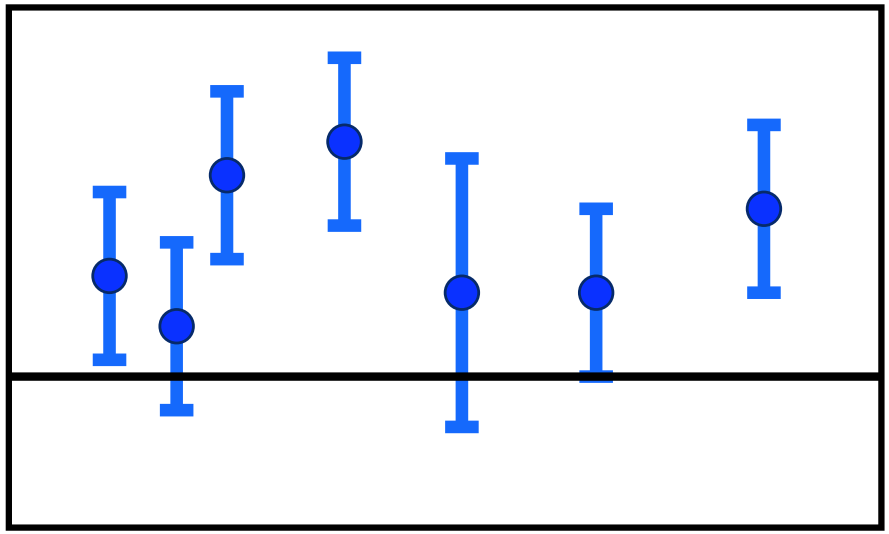
$$m(f) \equiv m_0 - 2.5 \log f$$

$$\sigma_m = (2.5 \log e) \frac{\sigma_f}{\langle f \rangle} \approx 1.08 \frac{\sigma_f}{\langle f \rangle}.$$

$$\langle m \rangle = m(\langle f \rangle) + \text{bias}(\langle m \rangle)$$

$$\text{bias}(\langle m \rangle) = \left(\frac{\sigma_m}{\sqrt{5 \log e}} \right)^2 \approx 0.01 \left(\frac{\sigma_m}{0.15} \right)^2$$

15% erorrs -> 1% bias
50% errors -> 10% bias



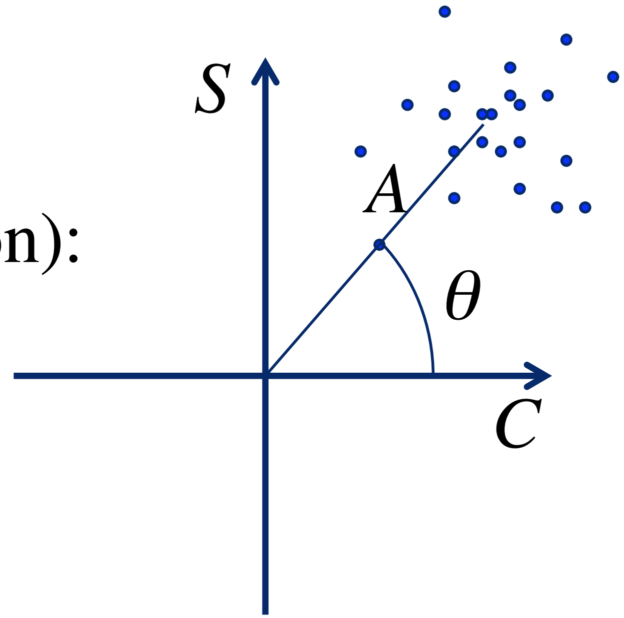
Given noisy fluxes, should you average fluxes and then compute the magnitude?

Or, should you convert fluxes to magnitudes and average the magnitudes ?

$$m(f) = m_0 - 2.5 \log \langle f \rangle$$

$$\text{or} \quad m(f) = \langle m_0 - 2.5 \log f \rangle$$

Cartesian -> Polar coordinates e.g. Amplitude and Phase



Independent measurements of C and S

(e.g. cos and sin amplitudes of an oscillation):

$$S = A\sin\theta \sim \left(S_0 \pm \sigma_S\right)$$

$$C = A\cos\theta \sim \left(C_0 \pm \sigma_C\right)$$

Transform to amplitude and phase:

$$A = ? \pm ?$$

$$\theta = ? \pm ?$$

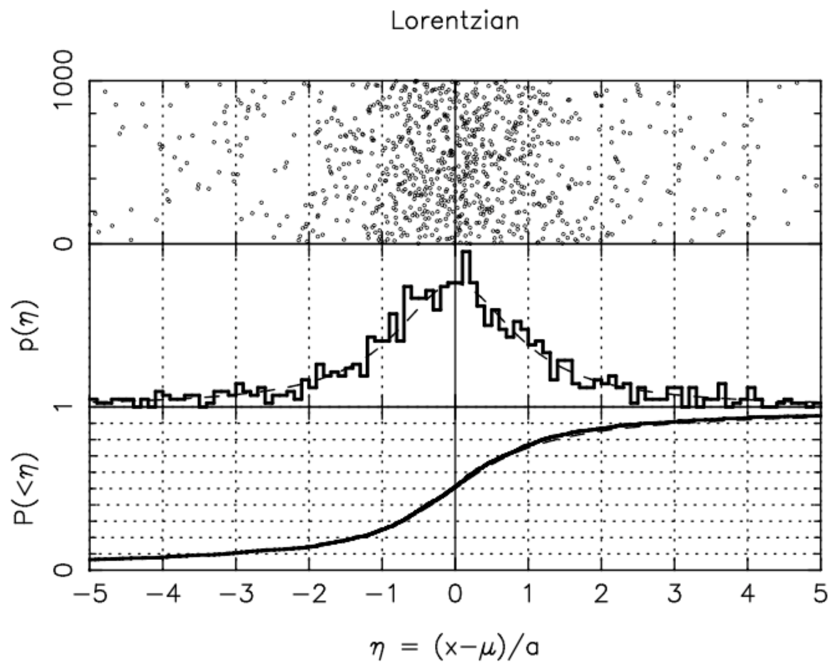
Transforming Random Numbers

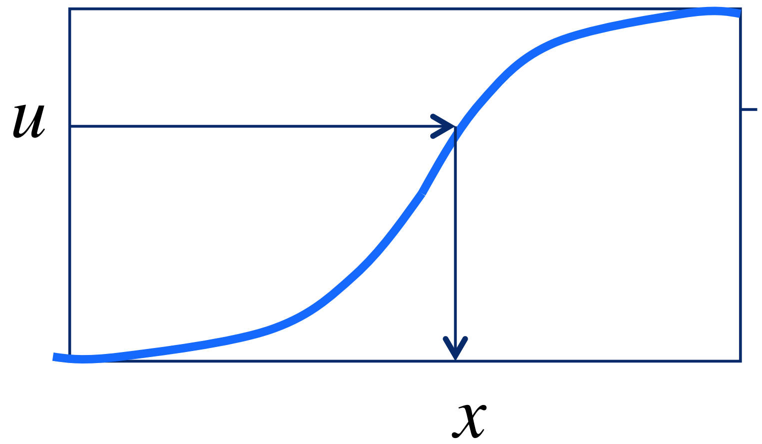
Uniform → Lorentzian

$u \sim U(0,1) \quad \rightarrow \quad x \sim L(\mu,\sigma)$

$$u \;=\; F(x) = \frac{1}{\pi}\arctan\left[\frac{x-\mu}{\sigma}\right] + \frac{1}{2}$$

$$x \;= F^{-1}(u) = \mu + \sigma \tan\left[\pi\left(u - \frac{1}{2}\right)\right]$$





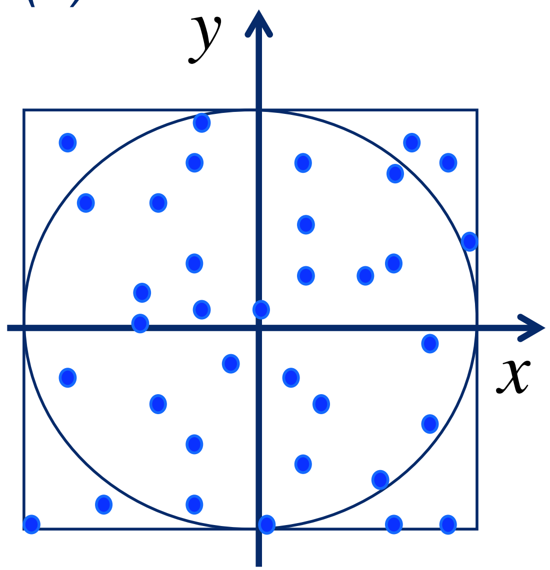
Practice :

Uniform → Exponential

Uniform → Power - law

Box-Muller Transform

For Gaussians, cumulative probability *F(x)* has no analytic expression.
Harder to generate Gaussian random numbers $x = F^{-1}(u)$
from Uniform random numbers *u*.



Two independent uniform random numbers :

$$x \sim U(-1, +1) \qquad y \sim U(-1, +1)$$

Keep if $\quad r^2 = x^2 + y^2 < 1$

Two independent gaussian random numbers :

$$G_1 = \frac{2\,x}{r}\left(-\ln r\right)^{1/2} \qquad G_2 = \frac{2\,y}{r}\left(-\ln r\right)^{1/2}$$

$$r = 0 \quad \rightarrow \quad G = \infty$$

$$r = 1 \quad \rightarrow \quad G = 0$$

G_1 and G_2 have mean 0 and variance 1:

$G_1 \sim G(0,1) \qquad G_2 \sim G(0,1)$