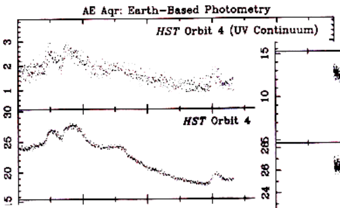
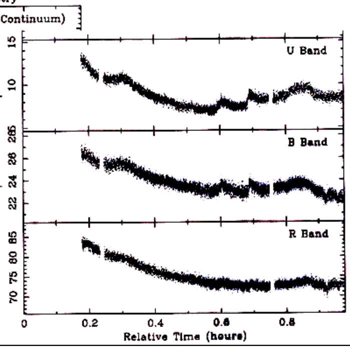
Eyeballing Sigma





Can you accurately estimate the standard deviation just by looking at a plot of the data ?

Eyeballing Sigma

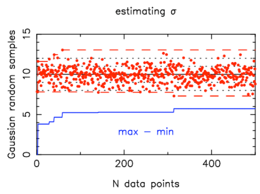
1. **If lots of data points: The 5-sigma rule:**

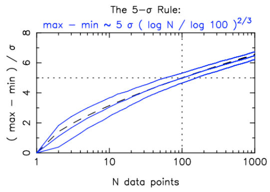
Estimate "by eye" the range (max-min) of the data (~100 data points).

That range (max-min) is about 5-sigma

2. **If only a few data points:**

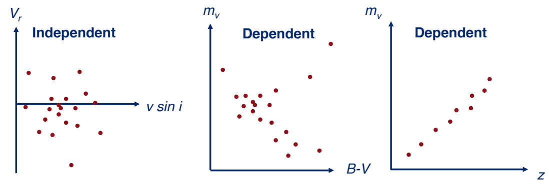
~1/3 of the data points should be outside +/- 1 sigma





Multivariate Distributions

- Suppose we measure 2 (or more) different properties
  - e.g. rotational and radial velocities of stars in a cluster
  - colours and magnitudes of stars in a cluster
  - redshifts and peak apparent magnitudes of distant supernovae

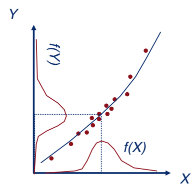


- Does knowing the value of one random variable X inform you about the other?

Joint Probability Distribution *f(X, Y)*

- *X* and *Y* are two random variables.
- Their **joint probability distribution** is *f(X, Y)*
- Normalisation: $\iint f(X,Y)\, dX\, dY = 1$
- Projection gives *f(X)*, *f(Y)* : $f(Y) = \int f(X,Y)\, dX$

$$f(X) = \int f(X,Y)\, dY$$



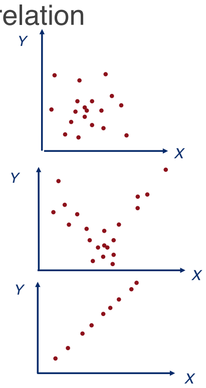
Independence and Correlation

- **Independence:**
  - knowing *X* does not inform about *Y*
  - Definition:

$$f(X,Y) = f(X)\, f(Y)$$

- **Partial correlation:**
  - knowing *X* tells you about *Y*

- **Correlation:**
  - *X* determines *Y*



The Algebra of Random Variables

Ordinary numbers are "sharp".

$$1 + 1 = 2$$

Random variables are "fuzzy" numbers.

$$(1 \pm 1) + (1 \pm 2) = (? \pm ?)$$

How do the mean and variance change when we add or subtract or multiply fuzzy numbers?

How do the higher moments change?

Linear Transformations: Scaling

Constants: $\langle a \rangle = ?$ $\quad \mathrm{Var}(a) = ?$

Scaling a random variable, *X*, by a constant, *a*:

- Mean:

$$\langle a X \rangle = a \langle X \rangle$$

$$\langle a X \rangle = \int f(X)\, a X \, dX = a \int f(X)\, X \, dX = a \langle X \rangle$$

- Variance:

$$\mathrm{Var}(aX) = a^2\, \mathrm{Var}(X)$$
$$\sigma(aX) = a\,\sigma(X)$$

$$\mathrm{Var}(aX) = \left\langle [aX - \langle aX \rangle]^2 \right\rangle = \left\langle [aX - a\langle X \rangle]^2 \right\rangle = \left\langle a^2 [X - \langle X \rangle]^2 \right\rangle = a^2\, \mathrm{Var}(X)$$

Linear Transformations: Addition

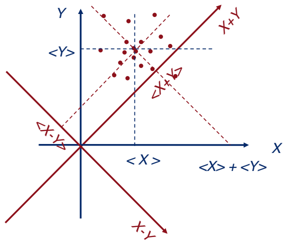
- Adding two random variables *X* and *Y*:

$$\langle X + Y \rangle = \iint f(X,Y)(X+Y)\, dX\, dY$$
$$= \iint f(X,Y) X\, dX\, dY + \iint f(X,Y) Y\, dX\, dY$$
$$= \int \left[\int f(X,Y)\, dY \right] X\, dX + \int \left[\int f(X,Y)\, dX \right] Y\, dY$$
$$= \int f(X) X\, dX + \int f(Y) Y\, dY$$
$$= \langle X \rangle + \langle Y \rangle$$

- True for **any** joint PDF! $\quad \langle X + Y \rangle = \langle X \rangle + \langle Y \rangle$

Why it works...

- **Centre of gravity** has a well-defined position, independent of choice of coordinates.
- e.g. could use either (*X, Y*) or (*X+Y, X-Y*) :



$$\langle X + Y \rangle = \langle X \rangle + \langle Y \rangle$$

Variance and Co-variance

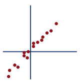
- Variance of *X+Y* depends on how *X* and *Y* co-vary:

$$\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\,\mathrm{Cov}(X,Y)$$
$$\mathrm{Cov}(X,Y) \equiv \left\langle (X - \langle X \rangle)(Y - \langle Y \rangle) \right\rangle$$

$$\mathrm{Var}(X+Y) = \left\langle [X + Y - \langle X+Y \rangle]^2 \right\rangle$$
$$= \left\langle [X + Y - \langle X \rangle - \langle Y \rangle]^2 \right\rangle$$
$$= \left\langle [(X - \langle X \rangle) + (Y - \langle Y \rangle)]^2 \right\rangle$$
$$= \left\langle (X - \langle X \rangle)^2 + (Y - \langle Y \rangle)^2 + 2(X - \langle X \rangle)(Y - \langle Y \rangle) \right\rangle$$
$$= \left\langle (X - \langle X \rangle)^2 \right\rangle + \left\langle (Y - \langle Y \rangle)^2 \right\rangle + 2\left\langle (X - \langle X \rangle)(Y - \langle Y \rangle) \right\rangle$$
$$= \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\,\mathrm{Cov}(X,Y)$$

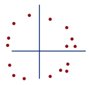
Co-variance

- **Cov > 0**

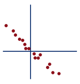


- **Cov = ?**

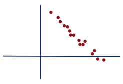
Independent?



- **Cov < 0**

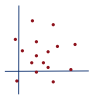


- **Cov = ?**

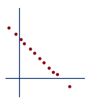


Practice !

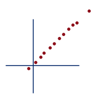
$X = 1 \pm 1$
$Y = 1 \pm 1$
$X + Y = ? \pm ?$



$X = 1 \pm 1$
$Y = 2 - X$
$X + Y = ? \pm ?$



$X = 1 \pm 1$
$Y = X$
$X + Y = ? \pm ?$



$X = 1 \pm 1 \quad Y = 2 \pm 1 \quad a = 2 \quad b = 1$

$Z = aX + bY \quad \langle Z \rangle = ? \quad \mathrm{Var}(Z) = ?$

Linear Transformations

- Scale and add any number of random variables:

$$\left\langle \sum_i a_i X_i \right\rangle = \sum_i a_i \langle X_i \rangle \qquad \mathrm{Var}\left[\sum_i a_i X_i\right] = \sum_{i,j} a_i a_j \mathrm{Cov}(X_i, X_j)$$

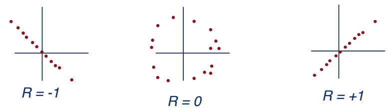
Or, in terms of the (symmetric) **Co-variance Matrix**:

$$\mathrm{Var}\left[\sum_i a_i X_i\right] = \sum_{i,j} a_i C_{ij} a_j$$

$$\mathrm{Var}\left[(a_i \;\; \dots \;\; a_N)\begin{pmatrix} X_1 \\ \dots \\ X_N \end{pmatrix}\right] = (a_i \;\; \dots \;\; a_N)\begin{pmatrix} C_{11} & \dots & C_{1N} \\ \dots & \dots & \dots \\ C_{N1} & \dots & C_{NN} \end{pmatrix}\begin{pmatrix} a_1 \\ \dots \\ a_N \end{pmatrix}$$

Correlation Coefficient *R(X, Y)*

- **Correlation coefficient**: $R(X,Y) \equiv \dfrac{\mathrm{Cov}(X,Y)}{\sigma(X)\sigma(Y)}$



- **Correlation matrix**: $$R_{ij} \equiv \frac{\mathrm{Cov}(X_i, X_j)}{\sigma(X_i)\,\sigma(X_j)} = \begin{bmatrix} 1 & . & . & . \\ . & 1 & . & . \\ . & . & 1 & . \\ . & . & . & 1 \end{bmatrix}$$

- **Variance**: $$\mathrm{Var}\left[\sum_i a_i X_i\right] = \sum_i \sum_j a_i a_j \sigma(X_i)\sigma(X_j) R_{ij}$$