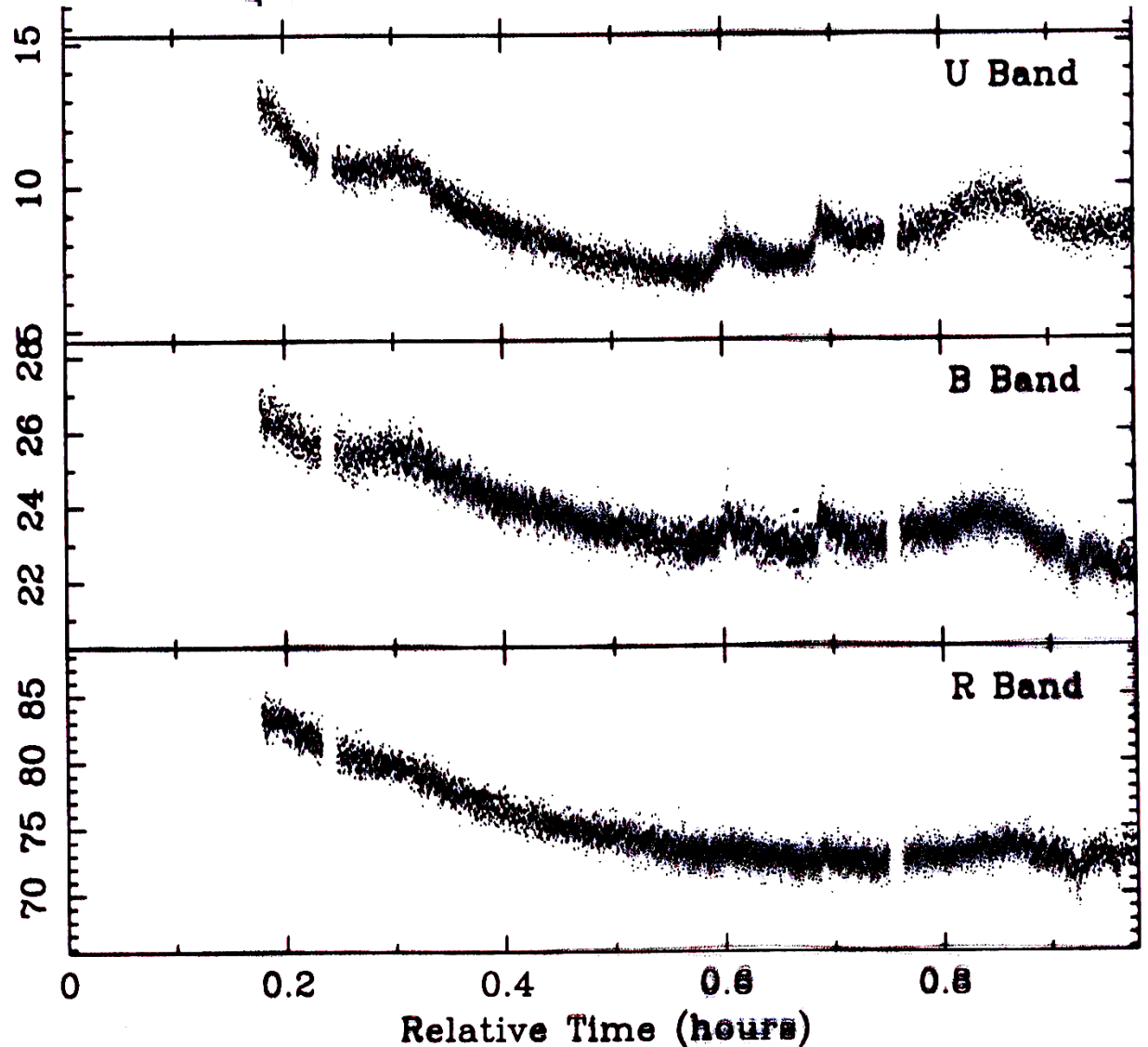
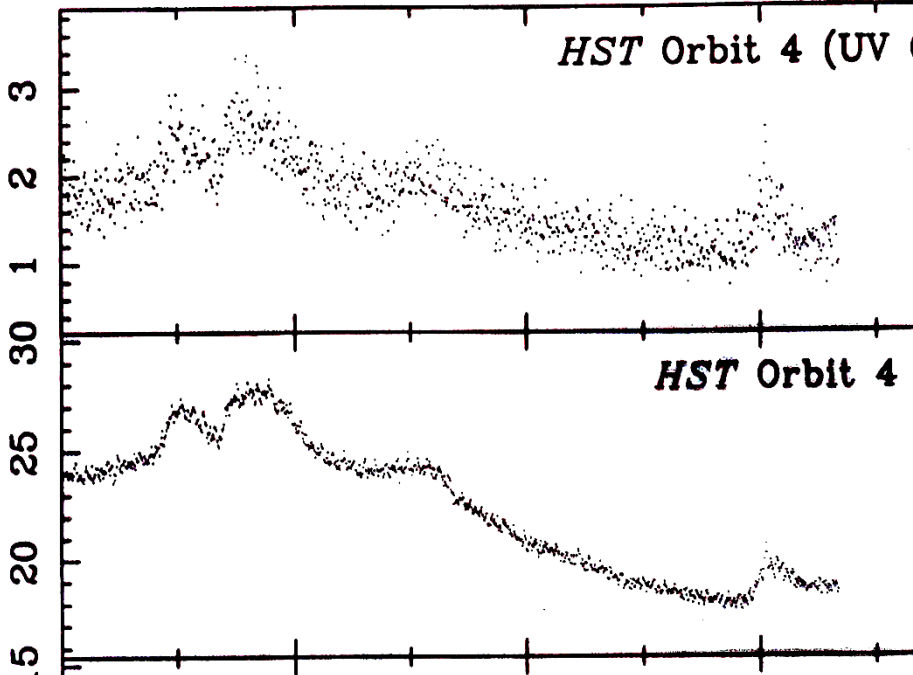


Eyeballing Sigma

AE Aqr: Earth-Based Photometry

HST Orbit 4 (UV Continuum)



Can you accurately estimate the standard deviation just by looking at a plot of the data ?

Eyeballing Sigma

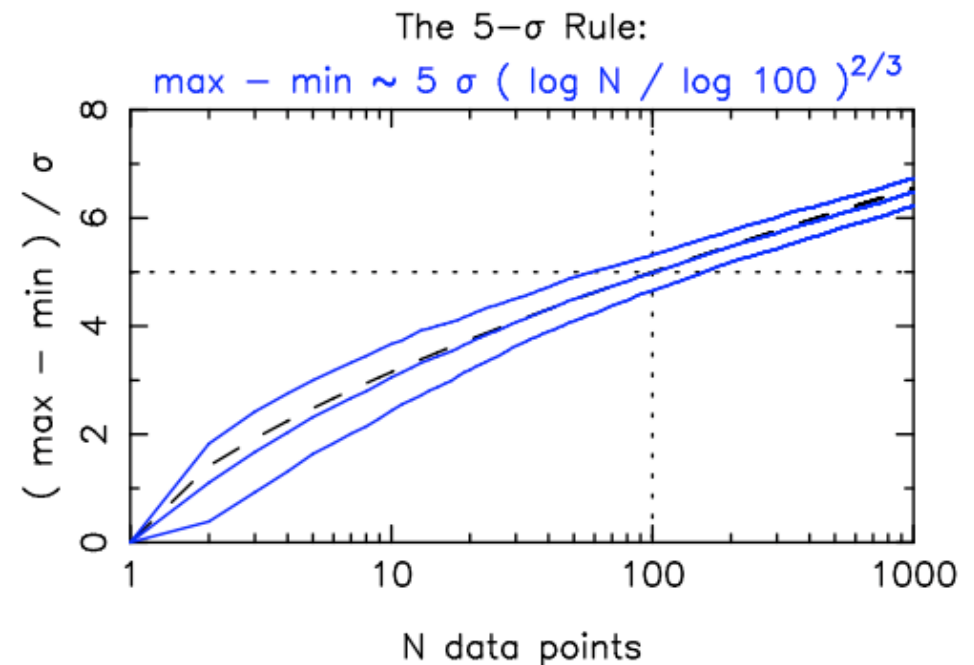
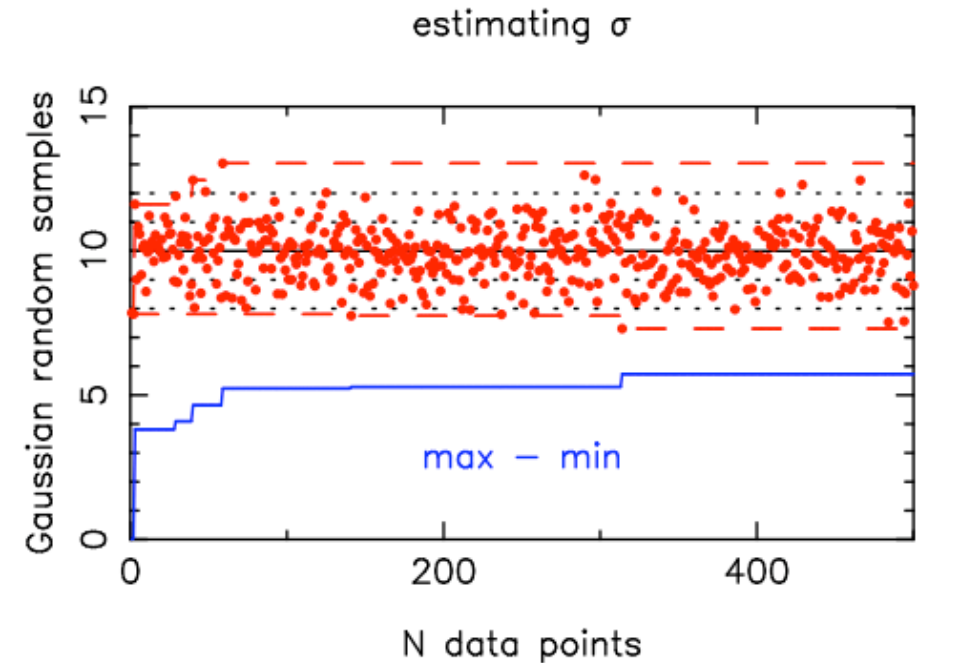
1. If lots of data points: The 5-sigma rule:

Estimate “by eye” the range (max-min) of the data (~100 data points).

That range (max-min) is about 5-sigma

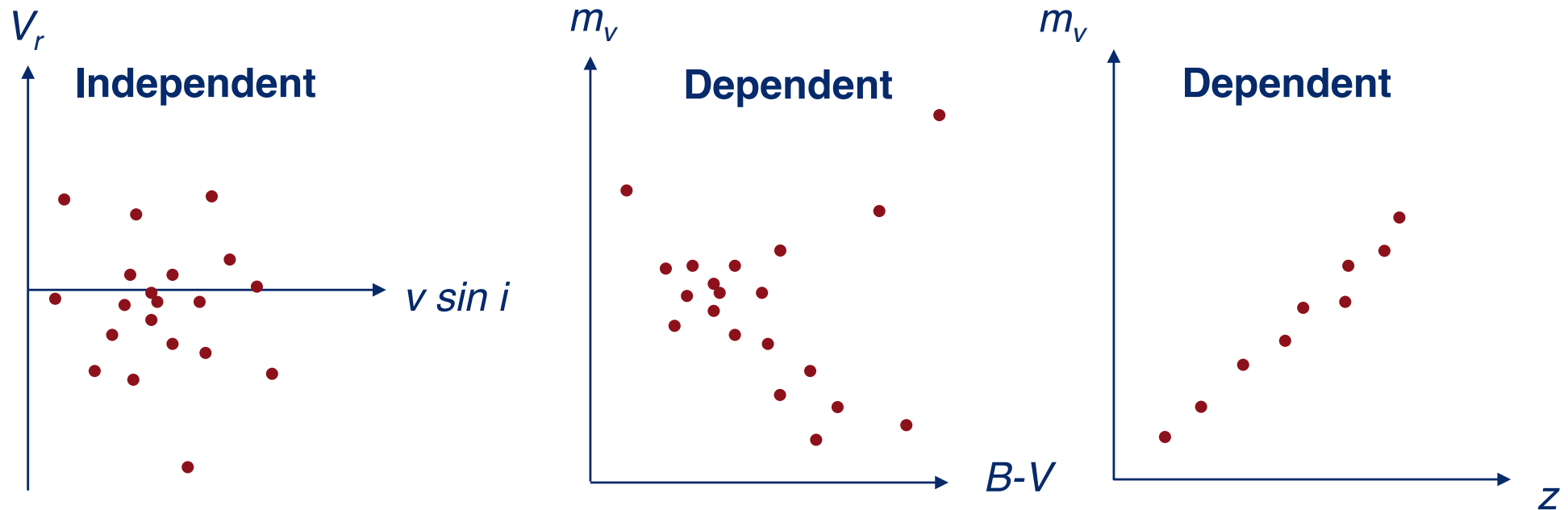
2. If only a few data points:

~1/3 of the data points should be outside +/- 1 sigma



Multivariate Distributions

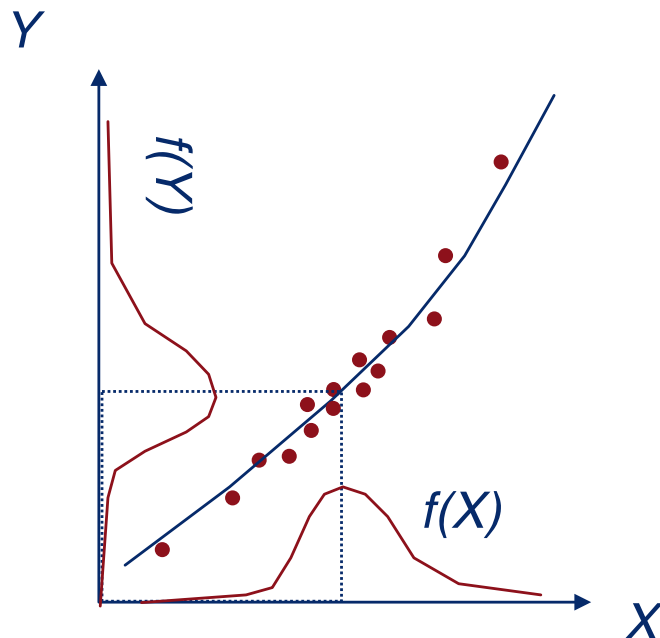
- Suppose we measure 2 (or more) different properties
 - e.g. rotational and radial velocities of stars in a cluster
 - colours and magnitudes of stars in a cluster
 - redshifts and peak apparent magnitudes of distant supernovae



- Does knowing the value of one random variable X inform you about the other?

Joint Probability Distribution $f(X, Y)$

- X and Y are two random variables.
- Their **joint probability distribution** is $f(X, Y)$
- Normalisation:
$$\iint f(X, Y) dX dY = 1$$
- Projection gives $f(X), f(Y)$:
$$f(Y) = \int f(X, Y) dX$$



$$f(X) = \int f(X, Y) dY$$

Independence and Correlation

- **Independence:**

- knowing X does not inform about Y
- Definition:

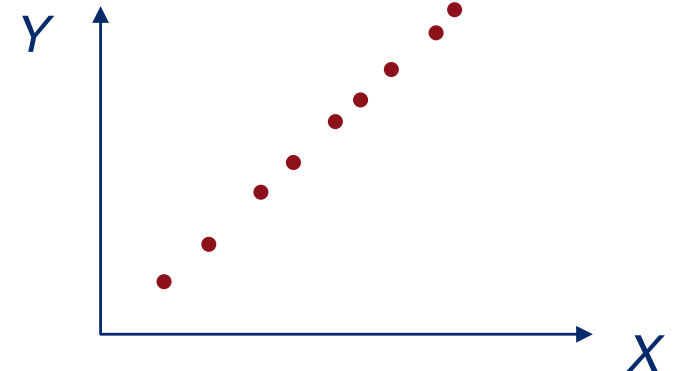
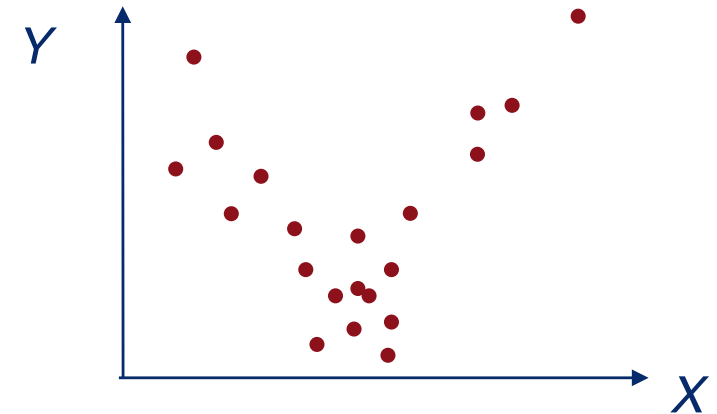
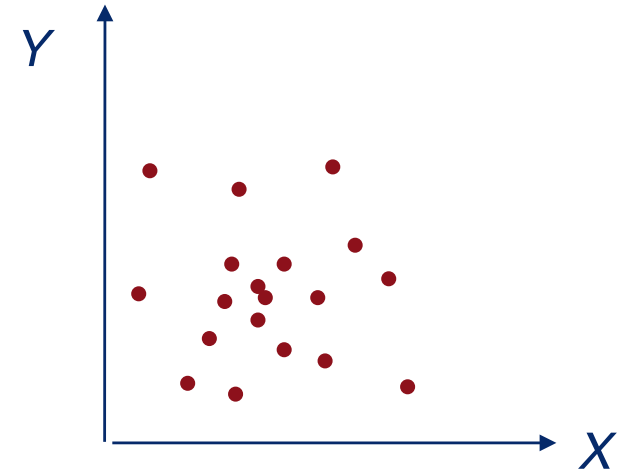
$$f(X, Y) = f(X) f(Y)$$

- **Partial correlation:**

- knowing X tells you about Y

- **Correlation:**

- X determines Y



The Algebra of Random Variables

Ordinary numbers are “sharp”.

$$1 + 1 = 2$$

Random variables are “fuzzy” numbers.

$$(1 \pm 1) + (1 \pm 2) = (? \pm ?)$$

How do the mean and variance change when we add or subtract or multiply fuzzy numbers?

How do the higher moments change?

Linear Transformations: Scaling

Constants: $\langle a \rangle = ?$ $\text{Var}(a) = ?$

Scaling a random variable, X , by a constant, a :

– **Mean:**

$$\langle aX \rangle = a\langle X \rangle$$

$$\begin{aligned}\langle aX \rangle &= \int f(X) aX dX \\ &= a \int f(X) X dX = a\langle X \rangle\end{aligned}$$

– **Variance:**

$$\begin{aligned}\text{Var}(aX) &= a^2 \text{Var}(X) \\ \sigma(aX) &= a\sigma(X)\end{aligned}$$

$$\begin{aligned}\text{Var}(aX) &= \langle [aX - \langle aX \rangle]^2 \rangle \\ &= \langle [aX - a\langle X \rangle]^2 \rangle \\ &= \langle a^2 [X - \langle X \rangle]^2 \rangle \\ &= a^2 \text{Var}(X)\end{aligned}$$

Linear Transformations: Addition

- Adding two random variables X and Y :

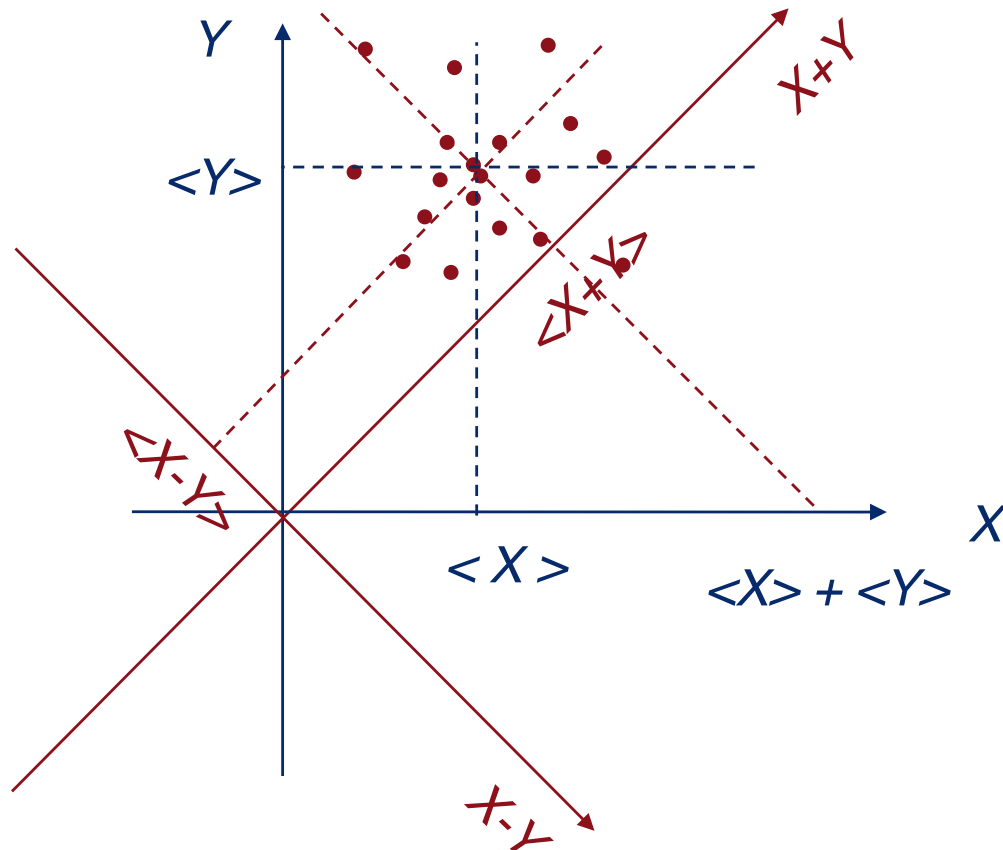
$$\begin{aligned}\langle X + Y \rangle &= \iint f(X, Y)(X + Y) dX dY \\ &= \iint f(X, Y)X dX dY + \iint f(X, Y)Y dX dY \\ &= \int \left[\int f(X, Y) dY \right] X dX + \int \left[\int f(X, Y) dX \right] Y dY \\ &= \int f(X)X dX + \int f(Y)Y dY \\ &= \langle X \rangle + \langle Y \rangle\end{aligned}$$

- True for **any** joint PDF!

$$\langle X + Y \rangle = \langle X \rangle + \langle Y \rangle$$

Why it works...

- **Centre of gravity** has a well-defined position, independent of choice of coordinates.
- e.g. could use either (X, Y) or $(X+Y, X-Y)$:



$$\langle X + Y \rangle = \langle X \rangle + \langle Y \rangle$$

Variance and Co-variance

- Variance of $X+Y$ depends on how X and Y co-vary:

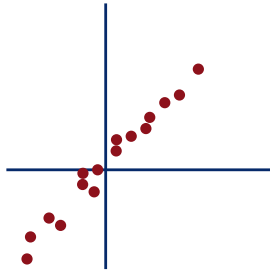
$$\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y) + 2\text{Cov}(X, Y)$$

$$\text{Cov}(X, Y) \equiv \langle (X - \langle X \rangle)(Y - \langle Y \rangle) \rangle$$

$$\begin{aligned}\text{Var}(X + Y) &\equiv \langle [X + Y - \langle X + Y \rangle]^2 \rangle \\ &= \langle [X + Y - \langle X \rangle - \langle Y \rangle]^2 \rangle \\ &= \langle [(X - \langle X \rangle) + (Y - \langle Y \rangle)]^2 \rangle \\ &= \langle (X - \langle X \rangle)^2 + (Y - \langle Y \rangle)^2 + 2(X - \langle X \rangle)(Y - \langle Y \rangle) \rangle \\ &= \langle (X - \langle X \rangle)^2 \rangle + \langle (Y - \langle Y \rangle)^2 \rangle + 2\langle (X - \langle X \rangle)(Y - \langle Y \rangle) \rangle \\ &= \text{Var}(X) + \text{Var}(Y) + 2\text{Cov}(X, Y)\end{aligned}$$

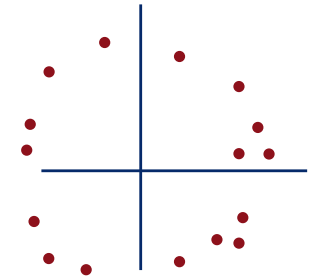
Co-variance

- **Cov > 0**

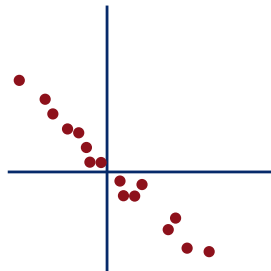


- **Cov = ?**

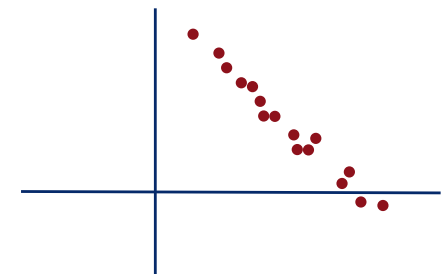
Independent?



- **Cov < 0**



- **Cov = ?**

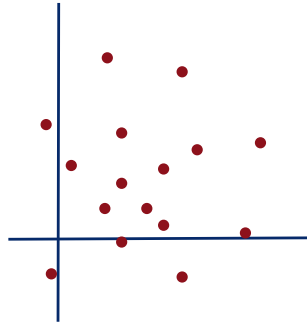


Practice !

$$X = 1 \pm 1$$

$$Y = 1 \pm 1$$

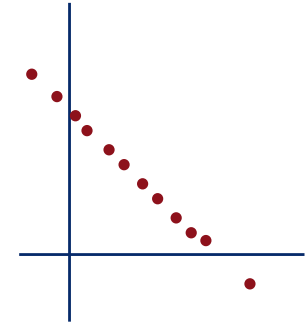
$$X + Y = ? \pm ?$$



$$X = 1 \pm 1$$

$$Y = 2 - X$$

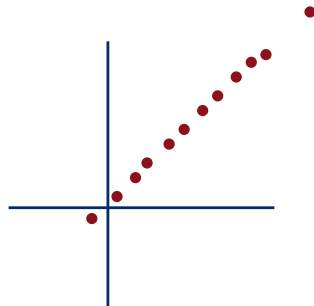
$$X + Y = ? \pm ?$$



$$X = 1 \pm 1$$

$$Y = X$$

$$X + Y = ? \pm ?$$



$$X = 1 \pm 1 \quad Y = 2 \pm 1 \quad a = 2 \quad b = 1$$

$$Z = aX + bY \quad \langle Z \rangle = ? \quad \text{Var}(Z) = ?$$

Linear Transformations

- Scale and add any number of random variables:

$$\left\langle \sum_i a_i X_i \right\rangle = \sum_i a_i \langle X_i \rangle \qquad \text{Var} \left[\sum_i a_i X_i \right] = \sum_{i,j} a_i a_j \text{Cov}(X_i, X_j)$$

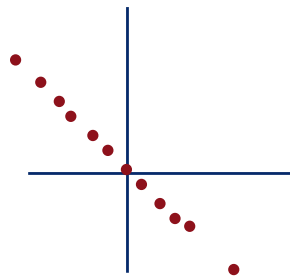
Or, in terms of the (symmetric) **Co-variance Matrix**:

$$\text{Var} \left[\sum_i a_i X_i \right] = \sum_{i,j} a_i C_{ij} a_j$$
$$\text{Var} \left[\begin{pmatrix} a_i & \dots & a_N \end{pmatrix} \begin{pmatrix} X_1 \\ \dots \\ X_N \end{pmatrix} \right] = \begin{pmatrix} a_i & \dots & a_N \end{pmatrix} \begin{pmatrix} C_{11} & \dots & C_{1N} \\ \dots & \dots & \dots \\ C_{N1} & \dots & C_{NN} \end{pmatrix} \begin{pmatrix} a_1 \\ \dots \\ a_N \end{pmatrix}$$

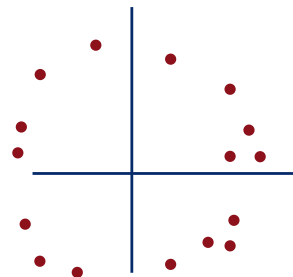
Correlation Coefficient $R(X, Y)$

- **Correlation coefficient:**

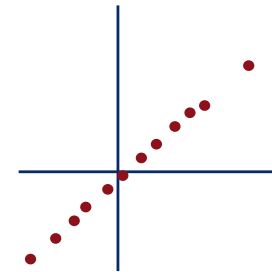
$$R(X, Y) \equiv \frac{\text{Cov}(X, Y)}{\sigma(X)\sigma(Y)}$$



$R = -1$



$R = 0$



$R = +1$

- **Correlation matrix:**

$$R_{ij} \equiv \frac{\text{Cov}(X_i, X_j)}{\sigma(X_i)\sigma(X_j)} = \begin{bmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \end{bmatrix}$$

- **Variance:**

$$\text{Var}\left[\sum_i a_i X_i\right] = \sum_i \sum_j a_i a_j \sigma(X_i) \sigma(X_j) R_{ij}$$