A relativistic theory of modified gravity has been recently proposed by Bekenstein. The tensor field in Einstein’s theory of gravity is replaced by a scalar, a vector, and a tensor field which interact in such a way to give Modified Newtonian Dynamics (MOND) in the weak-field non-relativistic limit. We study the evolution of the universe in such a theory, identifying its key properties and comparing it with the standard cosmology obtained in Einstein gravity. The evolution of the scalar field is akin to that of tracker fields in quintessence models. We expand the theory to linear order to find the evolution of perturbations on large scales. The impact on galaxy distributions and the cosmic microwave background is calculated in detail. We show that it may be possible to reproduce observations of the cosmic microwave background and galaxy distributions with Bekenstein’s theory of MOND.

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The current model of the Universe is based on a few simple assumptions and can explain a multitude of observation. Yet, to be able to explain the structure of galaxies and clusters of galaxies, it is essential to postulate the existence of some invisible substance, called dark matter. Although there are reports of tentative discoveries of dark matter [1], there is no proven theory or direct observation of a dark matter particle as yet. A less ex-

1. For \( \alpha \leq 0 \) we have that \( \mu(x) = 1 \). With \( |\alpha| < \alpha_0 \) we have that \( \mu(x) \approx x \). Clearly for small accelerations, Newtonian theory is no longer valid. Milgrom’s theory has been extremely successful in explaining a number of observational properties of galaxies. It has suffered from a fatal flaw in that it is not generally covariant and hence cannot be studied in a general setting.

Bekenstein has recently solved this problem [3]. Building on a series of developments [4], he has proposed a generally covariant theory which has in the non-relativistic, weak field limit, Milgrom’s modified theory. Bekenstein’s theory has two metrics. One of the metrics, \( \bar{g}_{\mu\nu} \), has its dynamics governed by the Einstein-Hilbert action,

\[
S_g = \frac{1}{16\pi G} \int d^4x \sqrt{-\bar{g}} \mathcal{R}
\]

where \( G \) is Newton’s constant and \( \mathcal{R} \) is the scalar curvature of \( \bar{g}_{\mu\nu} \). We shall call the frame of this metric the “Einstein Frame” (EF). The second metric, \( g_{\mu\nu} \) is minimally coupled to all the matter fields in the Universe. We shall call the frame of this metric the “Matter Frame” (MF). All geodesics are calculated in terms of this second metric. The two metrics are related through

\[
g_{\mu\nu} = e^{-2\phi}(\bar{g}_{\mu\nu} + A_\mu A_\nu) - e^{2\phi}A_\mu A_\nu.
\]

Two fields are required to connect the two metrics. The scalar field, \( \phi \) has dynamics given by the action

\[
S_\phi = -\frac{1}{16\pi G} \int d^4x \sqrt{-g} \left[ \mathcal{F}_{\mu\nu}^\alpha \mathcal{F}_{\alpha\beta} - 2\lambda (A^\mu A_\mu + 1) \right]
\]

where \( \mathcal{F}_{\mu\nu} = F_{\mu\nu} - A_{\mu}A_{\nu} \), indices are raised with \( \bar{g} \) and where \( K \) is the third parameter in this theory. The Lagrange multiplier \( \lambda \) is completely fixed by variation of the action.

We wish to study the evolution of a homogenous and isotropic Universe in such a theory. Observers are defined in the MF where the line element is \( ds^2 = a^2(-dt^2 + dx^2) \). The scale factor is related to the metric in the EF, \( ds^2 = a^2(-e^{-2\phi}dt^2 + dx^2) \) through \( a = e^{-\phi} \). The modified Friedmann equation becomes

\[
3\frac{\dot{a}^2}{a^2} = a^2 \left[ \frac{1}{2} e^{-2\phi} (\dot{\phi}^2 + V) + 8\pi G e^{-4\phi} \rho \right]
\]

where \( \dot{\phi} \) can be found by inverting \( \dot{\phi}^2 = \frac{1}{2} a^2 e^{-2\phi} \frac{dV}{d\phi} \) and the energy density \( \rho \) does not include \( \phi \). Homogeneity and isotropy and the constraint in the action imply that \( A_\mu \) is fixed as \( A_\mu = -\phi e^{-\phi}(0,0,0,0) \). The background dynamics is complete with an equation for \( \phi \):
density in one finds that for a wide range of initial conditions, the scalar field tracks the dominant form of energy at each instance in time.

\[ \dot{\phi} = -a^2 e^{-2\phi} V' - \frac{1}{U} \left[ 2\left( \frac{\mu}{V'} \right) \frac{\dot{b}}{b} + 4\pi G a^2 e^{-4\phi} (\rho + 3P) \right] \]

where \( U = \mu + 2V' / V'' \) and, once again, \( P \) does not include the pressure from the scalar field \( \phi \).

A choice of \( V \) will pick out a given theory. As a first guess, Bekenstein has proposed

\[ V = \frac{3\mu_0^2}{128\pi \ell_B^2} \left[ \tilde{\mu}(4 + 2\tilde{\mu} - 4\tilde{\mu}^2 + 2\ln(1 - \tilde{\mu})^2) \right] \]

where \( \tilde{\mu} = \mu / \mu_0 \). This potential will lead to the prescription proposed by Milgrom in the non-relativistic region. The evolution of this coupled set of equations is essentially insensitive to \( \ell_B \) and independent of \( K \) but is well determined in terms of \( \mu_0 \). If we define the physical Hubble parameter, \( H \) we can rewrite the modified Friedmann equations in the form \( 3H^2 = 8\pi G\rho_{\text{eff}}(\rho + \rho_\phi) \) where the effective Newton’s constant has the form \( G_{\text{eff}} = G e^{-4\phi} / (1 + \frac{\delta_\phi}{\alpha_{\text{eff}}})^2 \) and the energy density in \( \phi \) is

\[ \rho_\phi = \frac{e^{2\phi}}{16\pi G} (\mu V' + V) . \]

This system exhibits the tracking behaviour witnessed in some scalar field theories of quintessence [5]. In particular one finds that for a wide range of initial conditions, \( \phi \) evolves to a slowly varying function of time and the relative energy density in \( \phi \) reaches an attractor solution of the form \( \Omega_\phi = 3/(2\mu_0) \) in the radiation era and \( \Omega_\phi = 1/(6\mu_0) \) in the matter and \( \Lambda \) eras.

In Figure 1 we solve the equations numerically for a low value of \( \mu_0 \) to illustrate the tracking behaviour of \( \phi \). As with other tracking systems we can constrain the energy density in \( \phi \) at nucleosynthesis [6]. The abundance of light elements is extremely sensitive to the expansion rate at 1 MeV and can, for example, be used to constrain the number of relativistic degrees of freedom at that time. Recent measurements of the \(^4\text{He} \) mass fraction and the deuterium abundance leads to a bound at energies of 1 MeV of \( \Omega_\phi < 0.045 \) at the 95% confidence level. This leads to \( \mu_0 > 33 \) and hence the scalar field will make up less than 0.5% of the total energy density during the matter and \( \Lambda \) dominated eras.

![FIG. 1. The relative energy densities in \( \phi \) (thick solid line), radiation (dotted line), matter (dashed line) and \( \Lambda \) (dot-dashed line) for \( \mu_0 = 5 \) as a function of the scale factor (\( a \) is in arbitrary units). Note that the energy density in the scalar field tracks the dominant form of energy at each instance in time.](image)

![FIG. 2. The effect of the MONDian parameters on the power of spectrum of the CMB. Top panel: \( \mu_0 = 200, \ell_B = 100 \text{Mpc} \) and \( K = 1 \) (solid), 0.1 (dotted) and 0.08 (dashed); Middle panel: \( \mu_0 = 200, K = 1 \) and \( \ell_B = 1000 \text{Mpc} \) (solid), 100Mpc (dotted) and 10Mpc (dashed); Bottom panel: \( K = 0.1, \ell_B = 100 \text{Mpc} \) and \( \mu_0 = 1000 \) (solid), 200 (dotted) and 150 (dashed).](image)
scalar components, \( \alpha \) and \( E \), of \( \alpha_\mu = (\Psi + \varphi, \vec{\alpha}) \) are given by \( \nabla^2\alpha \equiv \nabla \cdot \vec{\alpha} \) and \( \nabla^2 E \equiv \nabla \cdot \vec{E} \) where we use the field strength tensor of \( A_\mu \), \( F_{\mu\nu} \) to define the “electric field” through \( E_\mu = \epsilon_{\mu \nu \lambda} F^{\nu \lambda} \). Similarly, in \( E \), we have \( \tilde{g}_{00} = -b^2 e^{-4\phi}(1 + 2\tilde{\Psi}) \), \( \tilde{g}_{0j} = -b^2 \tilde{\zeta}_i \), and \( \tilde{g}_{ij} = b^2(1 + 2\Phi)\delta_{ij} \) which give \( \Psi = \Psi + \varphi \), \( \tilde{\Phi} = \Phi + \varphi \) and \( \tilde{\zeta} = -(1 - e^{-4\phi})\alpha \).

The evolution equations for the matter fluid remain unaltered if expressed in terms of the MF variables. That is if we expand densities as \( \rho = \bar{\rho}(1 + \delta) \) and use the standard definition for momentum of the fluid, \( \theta = \nabla \cdot \vec{\eta} \), the evolution equations remain the same as in Einstein gravity. Two new sets of evolution equations must be introduced. For the scalar field perturbations, we have

\[
\dot{\varphi} = -\frac{ae^{-\phi}}{2U} \gamma + \tilde{\gamma} \psi
\]

\[
\dot{\gamma} = -3\frac{b}{b_0} + \frac{\tilde{\rho}}{a} e^{-4\phi}(k_0^2 \kappa^2 - 2\frac{b}{a} \phi) + 2\frac{e^{-\phi}}{a} (3\dot{\Phi} + k^2 \tilde{\zeta}) + 8\pi Gae^{-3\phi} \bar{\rho} \left( 1 + c_2^2 \delta (1 + 3w)(\Psi - 2\varphi) \right)
\]

and for the vector field we have

\[
\dot{\alpha} = \frac{e^{-\phi}}{a} E \Psi + (\varphi - \dot{\phi} - \alpha) \alpha
\]

\[
K \frac{e^{2\phi}}{a^2} (\dot{E} + 2\dot{\phi} E) = -\mu e^{-\phi} \left( \varphi - \dot{\phi} \alpha \right) + 16\pi Gae^{-\phi} \sinh(2\phi)(1 + w) \bar{\rho} (\theta - \alpha)
\]

where \( w = \frac{P}{\bar{\rho}} \) and \( c_2^2 = \delta P/\delta \bar{\rho} \). The perturbed Einstein equations allow us to identify the gravitational potentials through:

\[
2k^2 \dot{\Phi} = -2e^{4\phi} \frac{b}{b_0} k^2 \zeta - e^{4\phi} \dot{\phi} \left\{ -ae^{-\phi} \gamma + 6\mu \frac{b}{b_0} \varphi \right\}
\]

\[
-8\pi G\bar{\rho} \left( \delta + 3(1 + w) \frac{b}{b_0} \theta - 2\varphi \right) - Kk^2 e^{\phi} \frac{1}{a} E
\]

\[
\dot{\tilde{\Phi}} = 4\pi Gae^{-4\phi}(1 + w) \bar{\rho} \theta + \tilde{\mu} \phi \varphi - \frac{b}{b_0} \tilde{\Psi}
\]

\[
\Psi = \tilde{\Phi} + \tilde{\zeta} + 2 \left( \frac{b}{b_0} + \phi \right) \zeta - \frac{12\pi G}{k^2} \alpha^2 (1 + w) \bar{\rho} \sigma.
\]

where \( \sigma \) is the total shear from matter fluids.

We have modified CMBEASY, a publicly available numerical Einstein-Boltzmann solver to incorporate the modified background and perturbation equations [9]. The evolution equations have been implemented in both the conformal Newtonian and synchronous gauge to check for consistency. We have restricted ourselves to a flat Universe with a cosmological constant but considered the possibility of massive neutrinos.

The parameters \( \mu_0 \), \( K \) and \( \ell_B \) may introduce major modifications in the morphology of perturbations. A low \( \mu_0 \), low \( \ell_B \) and low \( K \) will lead to a change in the growth rate. As we can see in Figure 2, the effect is to introduce an integrated Sachs Wolfe term which can be quite significant. For example, for sufficiently small \( \ell_B \), the structure of the angular power spectrum of the CMB can be completely modified with an excess of large scale power overwhelming structure on the smallest scales. We can see the effect of modifying \( \ell_B \) in the lower panel of Figure 2. To remove this effect one has to take the appropriate joint limit of \( K, \ell_B \) and \( \mu_0 \) with \( K \propto 1/\mu_0 \) and \( \ell_B \propto \mu_0^{3/2} \) as \( \mu_0 \to \infty \). Clearly, the CMB can place quite stringent constraints on the values of these parameters.

Since the baryon content is set by the abundance of light elements, we must compensate with a high value of the cosmological constant, i.e. with \( \Omega_\Lambda \simeq 0.95 \). An obvious consequence of this is that the angular-distance relation will be modified as compared to the standard adiabatic ΛCDM universe [10]. Indeed the position of the peaks in the angular power spectrum of the CMB will be shifted to higher \( \ell_6 \) which would lead to a severe mismatch with the current available data from WMAP and other experiments. A natural solution to this is to include a small component of massive neutrinos, \( \Omega_\nu \simeq 0.15 \). As we can see in the top panel of Figure 4, with this modification we can reproduce the temperature anisotropy data.

The main question we have raised is whether MONDian dynamics can inhibit the damping of small scale perturbations in the coupled baryon-photon fluid during recombination. Recall that in the adiabatic CDM
As mentioned above, a universe with a very large but also overcoming damping on small scale. In Figure 4, the angular power spectrum of the CMB and galaxy surveys? There is clearly a competition between overproducing large scale power in the CMB and galaxy surveys? It is possible to reproduce current observations of the structure, is it possible to construct a MONDian universe which is roughly given by $k^2 \Phi \simeq 4\pi G(\rho_B \delta_B + \rho_C \delta_C)$ will not be erased if $\rho_C$ is sufficiently large, even though $\delta_B \rightarrow 0$ through recombination [11]. In the MONDian universe we find an analogous effect; we now have $k^2 \Phi \simeq 4\pi G(\rho_B \delta_B - 2\rho_0 \delta_C)$. The perturbation in the scalar field will support the perturbations through recombination. Unlike the case of dark matter however, the coupling between the scalar field and the metric is such that $\rho_0$ does not play a role in the magnitude of the effect. Even for minute values of $\Omega_0$ we can still have a non-negligible effect. As we can see in Figure 3, the net result is that decreasing $\mu_0$, $\ell_B$ or $K$ will boost small scale power in such a way as to overcome the damping of perturbations. This is an intriguing effect that goes in tandem with what we saw in the CMB. While decreasing $\ell_B$ (and a sufficiently small $K$ and $\mu_0$) will contaminate the large scale power in the angular power spectrum of the CMB, it can also play a role in counteracting Silk damping of density perturbations.

Given these two effects on the dynamics of large scale structure, is it possible to construct a MONDian universe which can reproduce current observations of the CMB and galaxy surveys? There is clearly a competition between overproducing large scale power in the CMB but also overcoming damping on small scale. In Figure 4 we present two MONDian universes compared to data [12,13]. As mentioned above, a universe with a very large contribution of $\Lambda$ will not fit the current CMB data. By having the three neutrinos with a mass of $m_\nu \simeq 2$ eV each we are able to resolve this mismatch. With an appropriate choice of $K$, $\mu_0$ and $\ell_B$ it is possible to reproduce the power spectrum of galaxies as inferred from the Sloan Digital Sky Survey [13]. The possibility of using massive neutrinos to resolve some of the problems with clusters in a MONDian universe has been mooted in [14].

We have focused on one very specific model proposed by Bekenstein with a somewhat artificial potential for the new degrees of freedom. This phenomenological approach needs a firmer theoretical underpinning which might come from the various approaches which are being taken in the context of brane worlds, M-theory and a rich array of theories of modified gravity. However, Bekenstein’s theory can play an important role in opening up an altogether different approach to the dark matter problem. It serves as a proof of concept which will clearly lead to a new, very different view of the role played by the gravitational field in cosmology.

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