Dynamics of non-spherically symmetric systems and N-body simulations in MOND

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OUTLINE

- The non-linear MOND field equation
- Analytical solutions:
  - The Kuzmin disk (Brada & Milgrom 1995)
  - New axisymmetric and triaxial models (Ciotti, Londrillo & Nipoti 2006)
- Numerical solutions:
  - A new numerical MOND potential solver (Ciotti, Londrillo & Nipoti 2006)
  - Testing the code
- Applications:
  - Estimating the solenoidal field in galaxy models
  - Vertical force in disks: MOND vs DM (Zhao, Nipoti, Londrillo & Ciotti in prep)
  - N-body simulations in MOND
- Conclusions
The non-linear MOND field equation

\[ \nabla \cdot \left[ \mu \left( \frac{ \| \nabla \phi \| }{a_0} \right) \nabla \phi \right] = 4 \pi G \rho \]

replaces the Poisson equation

\[ a_0 = 1.2 \times 10^{-10} \text{ m s}^{-2} \]

characteristic acceleration

\[ \ddot{g} = -\nabla \phi \]

gravitational field

\[ \mu(x) = \frac{x}{\sqrt{1 + x^2}} \]

the \( \mu \) function

(Bekenstein & Milgrom 1984)

Deep MOND regime
(low surface density systems)

\[ g \ll a_0 \Rightarrow \mu \approx 1 \Rightarrow \nabla^2 \phi_N = 4 \pi G \rho \]

Newtonian regime
(high surface density systems)

\[ g \gg a_0 \Rightarrow \mu \approx x \Rightarrow \nabla \cdot (\| \nabla \phi \| \nabla \phi) = 4 \pi G \rho a_0 \]

Globular cluster 47tuc

LSB NGC1560
The solenoidal field $S = \text{curl } h$

The MOND potential $\phi$ is related to the Newtonian potential $\phi_N$ by

$$\mu \left( \frac{\| \nabla \phi \|}{a_0} \right) \nabla \phi = \nabla \phi_N + \tilde{S}$$

$\tilde{S} = \nabla \times \tilde{h}$ is an unknown solenoidal field

Only in case of for spherical, cylindrical, planar symmetry $\tilde{S} = 0$

$\Rightarrow \mu \nabla \phi = \nabla \phi_N$ easy algebraic solution (Milgrom 1983 empirical formula)

In general $\tilde{S} \neq 0$ and one has to solve the non-linear field equation

$$\nabla \cdot \left[ \mu \left( \frac{\| \nabla \phi \|}{a_0} \right) \nabla \phi \right] = 4\pi G \rho$$

Also for axisymmetric systems!!!

(Bekenstein & Milgrom 1984)
An exception: the Kuzmin disk
(Brada & Milgrom 1995)

The razor-thin Kuzmin disk is the ONLY known axisymmetric model for which \( \hat{S} = 0 \) because
\[
\mathcal{S} = \left| \nabla \phi_N \right| = g_N(\phi_N)
\]

\[
\rho_{\text{Kuzmin}}(R) = \frac{aM}{2\pi \left( R^2 + a^2 \right)^{3/2}}
\]

The MOND potential of the Kuzmin disk is known \textit{analytically}
- Useful to test MOND out of spherical symmetry
- BUT quite unrealistic as a galaxy model!

For instance, in deep MOND:
\[
\phi_{\text{Kuzmin}}(R, z) = \sqrt{GMa_0 \ln \left( R^2 + (|z| + a)^2 \right)^{1/2}}
\]

\( \rho_{\text{Kuzmin}}(R, z) \)
\( \phi_{\text{Kuzmin}}(R, z) \)
We propose a **general** method to build analytical axisymmetric and triaxial density–potential pairs

ϕ-to-ρ approach: deformation of a spherically symmetric solution

1) Choose a spherical density and compute the MOND potential:

\[ \rho_0(r) \Rightarrow \phi_0(r) \]

2) Add an **aspherical** function to the potential and compute the corresponding density using the MOND field equation:

\[ \phi(r, \theta, \varphi) \equiv \phi_0(r) + \lambda \phi_1(r, \theta, \varphi) \Rightarrow \rho(r, \theta, \varphi) \]

3) If the density is everywhere positive \((\phi, \rho)\) is an aspherical MOND density–potential pair

**FOR A SUITABLE CHOICE OF \(\phi_1\) AND SMALL ENOUGH \(\lambda\) A POSITIVE DENSITY IS FOUND**
ANALYTICAL SOLUTIONS

An example: analytical axisymmetric and triaxial Hernquist models in MOND

- Analytical density & potential (+)
- General method (+)
- Realistic density profile (+)
- Significant flattening \((0.6 < b/a < 1)\) (+)

- Not highly flattened systems (-)
- Density is not 100% under control (-)

\[
\rho_0(r) = \frac{M}{2\pi} \frac{1}{r(1+r)^3}
\]

\[
\phi_1(r, \theta) \propto \frac{r \cos^2 \theta}{(r+1)^2}
\]
A new numerical MOND potential solver
(Ciotti, Londrillo & Nipoti 2006, Apj)

- We developed a new numerical solver for the non-linear MOND field equation
- Non-linear elliptic equations → relaxation method → Newton iterative method
- Spherical coordinates grid ($N_r, N_\theta, N_\phi$)
- Spectral method in angular variables (spherical harmonics)
- Finite differences in radial coordinate
- The solver can be used in particle-mesh N-body codes (e.g. Londrillo & Messina 1990)
- Designed for finite-mass, single-peaked density distributions

→ Literature: very little work on numerical solution of the MOND field equation
  (Brada & Milgrom 1995, 1999: Cartesian coordinates + multigrid method)
The numerical method

**Newton iteration**

Exact operator:
- Quadratic convergence
- Inversion of a 3-D matrix required
- Numerical difficulties

**INEFFICIENT!**

Approximate operator:
- Only Linear convergence
- Exploits spherical harmonics
- Inversion of a 1-D matrix

**YES!**

At each iteration step, one radial equation for each \((l,m)\) component

At each iteration step, one radial equation for each \((l,m)\) component

\[
\hat{M}[\phi(x)] = \nabla \left[ \mu \left( \frac{g}{a_0} \right) \nabla \phi(x) \right] + 4\pi G \rho(x) = 0, \quad g = O(r^{-1}) \text{ for } r \to \infty
\]

\[
\phi^{(n+1)} = \phi^{(n)} + \delta \phi^{(n)}
\]

\[
\delta \hat{M}^{(n)} \left[ \delta \phi^{(n)} \right] = -\hat{M} \left[ \phi^{(n)} \right]
\]

\[
\delta \hat{M}^{(n)} = \nabla \cdot \left[ \mu^{(n)} \nabla + \mu^{(n)} \hat{g}^{(n)} \left( \hat{g}^{(n)} \cdot \nabla \right) \right]
\]

\[
\hat{M} \left[ \phi^{(n+1)} \right] - \hat{M} \left[ \phi^{(n)} \right] = \delta \hat{M}^{(n)} \left[ \delta \phi^{(n)} \right] + O \left[ (\delta \phi^{(n)})^2 \right]
\]

\[
\delta \hat{M}^{(n)} = \frac{1}{r^2} \left[ \frac{\partial}{\partial r} \left( r^2 \bar{\mu}^{(n)}(r) \frac{\partial}{\partial r} \right) + \bar{\mu}^{(n)}(r) \left( \hat{L}_\theta + \hat{L}_\phi \right) \right]
\]

\[
\hat{L}_\theta \equiv \frac{1}{\sin \vartheta} \frac{\partial}{\partial \vartheta} \left( \sin \vartheta \frac{\partial}{\partial \vartheta} \right), \quad \hat{L}_\phi \equiv \frac{1}{\sin \vartheta} \frac{\partial^2}{\partial \varphi^2}
\]

\[
\bar{\mu}^{(n)}(r) = (1/4\pi) \int \mu^{(n)}(r, \vartheta, \varphi) \sin \vartheta d\vartheta d\varphi
\]

\[
\hat{M} \left[ \phi^{(n+1)} \right] - \hat{M} \left[ \phi^{(n)} \right] = \delta \hat{M}^{(n)} \left[ \delta \phi^{(n)} \right] + O \left[ \delta \phi^{(n)} \right]
\]

\[
\delta \phi^{(n)}(r, \vartheta, \varphi) = \sum_{l,m} \delta \phi_{lm}^{(n)}(r) Y_{lm}(\vartheta, \varphi)
\]

\[
\frac{1}{r^2} \left[ \frac{\partial}{\partial r} \left( r^2 \bar{\mu}^{(n)} \frac{\partial}{\partial r} \right) - \bar{\mu}^{(n)}(r) l(l + 1) \right] \delta \phi_{lm}^{(n)}(r) = -\hat{M} \left[ \phi^{(n)} \right]_{lm}
\]
Comparison with non-spherical analytical solutions:

- Kuzmin disk
- Triaxial Hernquist models
APPLICATIONS: ESTIMATING THE SOLENOIDAL FIELD

HOW IMPORTANT IS THE SOLENOIDAL FIELD $S$?

We used the numerical solver to estimate the solenoidal field $S$ in astrophysically relevant systems.

- **S** is typically small compared to the MOND acceleration $g$ ($S/g < 0.1$) (in agreement with Brada & Milgrom 1995).
- **This is not always true:** in deep MOND systems (e.g. low-surface density axisymmetric Hernquist models) $s/g$ is as high as 0.6 at the centre!

\[
\hat{S} = \mu \nabla \phi - \nabla \phi_N
\]

MOND rotation curve for an exponential disk

Yellow: exact
Red: assume $S=0$

\[
\frac{\hat{S}}{||\nabla \phi||}
\]

for an axysymmetric Hernquist model
APPLICATIONS: DISK VERTICAL FORCE: MOND vs DARK MATTER

VERTICAL FORCE IN DISK GALAXIES IN MOND AND DM

*Preliminary results* (Zhao, Nipoti, Londrillo & Ciotti in prep)

Given a the surface density of baryons in a disk galaxy, **MOND predicts the vertical force field**.

For the same galaxy, in a DM scenario the **disk+(spherical)halo model** reproducing the rotation curve **predicts a different vertical force field**.

Good measures of the **vertical velocity dispersion** of observed disk galaxies can discriminate between the two scenarios.

**NUMERICAL SOLUTION**
- Exponential disk: \(M = 10^{10}\) Msun
  - \(h_R = 3\) kpc, \(h_z = 0.3\) kpc

**ANALYTICAL SOLUTION**
- Kuzmin disk: \(M = 10^{10}\) Msun
  - \(h_R = 2.5\) kpc

Both MOND and disk+DM halo reproduce the **same rotation curve** but **MOND predicts higher vertical velocity dispersion than disk+dark matter halo Newtonian gravity**.

Application to observational data:
- **MILKY WAY** (vertical vel disp. in the solar neighborhood) / other galaxies
APPLICATIONS: N-BODY SIMULATIONS IN MOND

N-body simulations in MOND

- No Green function --> NO TREECODE, NO DIRECT N-BODY CODE
- Instead use --> PARTICLE-MESH CODES
- WE CANNOT NEGLECT THE SOLENOIDAL FIELD S
  (even if S is typically small in stationary systems!)
- WE MUST SOLVE EXACTLY THE FIELD EQUATION

If one neglects the solenoidal field S, momentum is not conserved
(Felten 1984, Bekenstein & Milgrom 1984)

The ONLY MOND N-body simulations so far were those by
Few applications: disk stability, external field effect
A new particle–mesh MOND N–body code

(Londrillo, Nipoti & Ciotti in preparation)

- We developed a new code to run N–body simulations in MOND
- Standard particle–mesh technique used in Newtonian codes
- The Poisson solver is replaced by our new MOND potential solver
- Standard leap–frog time integration
- The code is (partly) parallel
APPLICATIONS: N-BODY SIMULATIONS IN MOND

Simulations of dissipationless collapse in MOND
(Nipoti, Londrillo & Ciotti in preparation)

- We ran simulations of cold collapse of a set of N-particles in MOND
- \( N = 1-2 \times 10^6 \) particles
- Initial conditions: clumpy, spherically symmetric Plummer distribution with particles at rest
- We check energy, linear and angular momentum conservation

**Time evolution**

\[ W = -\int \rho (\vec{x}) \langle \vec{x}, \nabla \phi \rangle d^3 \vec{x} \]

Virial Theorem holds in MOND
(Gerhard & Spergel 1992)

- W is conserved in deep MOND. This can be proved analytically (Nipoti et al. in prep).
- Conservation of total energy \( K + W \) in Newtonian gravity
APPLICATIONS: N-BODY SIMULATIONS IN MOND

End-products of dissipationless collapse in MOND

Newtonian collapses: $R^{1/4}$ de Vaucouleurs profiles (see van Albada 1982) well reproduced by our code

MOND collapses produce systems with shallower inner cusps: Sersic $R^{1/m}$ profile with $m=2-3$

MOND end-products have flatter velocity dispersion profile and are more radially anisotropic than Newtonian end-products
SUMMARY & CONCLUSIONS

ANALYTICAL METHODS AND NUMERICAL CODES

- We presented a flexible method to build analytical axisymmetric and triaxial MOND density-potential pairs with realistic density distributions
- We developed and tested a numerical MOND potential solver for generic density distributions
- We developed and tested a parallel particle-mesh code for MOND N-body simulations

APPLICATIONS AND FIRST RESULTS

- The (often neglected) solenoidal field $S$ is typically small in stationary systems, BUT in some (low-surface density) systems we found $S/g$ up to 0.6
- Preliminary results of N-body simulations suggest that the end-products of cold collapse in MOND differ structurally and kinematically from the end-products of Newtonian collapse

WORK IN PROGRESS & FUTURE APPLICATIONS

- Vertical kinematics of disk galaxies in MOND
- Constraints on the $\mu$ function from rotation curves
- TeVeS gravitational lensing from non-spherical lenses
- Stability of disks in MOND

(Also in collaboration with P. Londrillo, L. Ciotti, H. Zhao & B. Famaey)

NOTE: here we considered the Bekenstein & Milgrom (1984) $\mu$ function but our numerical code works for all the proposed $\mu$ functions for MOND and TeVeS