Gravitational Dynamics: Part II

Non-Equilibrium systems

Lec12: Growth of a Black Hole by capturing objects in Loss Cone

- A small BH on orbit with pericentre $r_p<R_{bh}$ is lost (as a whole) in the bigger BH.
  - The final process is at relativistic speed. Newtonian theory is not adequate
- (Nearly radial) orbits with angular momentum $J<J_{lc}=2*c*R_{bh}=4GM_{bh}/c$ enters `loss cone` (lc)
- When two BHs merger, the new BH has a mass somewhat less than the sum, due to gravitational radiation.

Size and Density of a BH

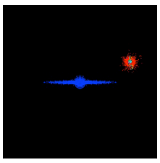
- A black hole has a finite (schwarzschild) radius $R_{bh}=2\,G\,M_{bh}/c^2 \sim 2\text{au}\,(M_{bh}/10^8M_{sun})$
  - verify this! What is the mass of 1cm BH?
- A BH has a density $(3/4Pi)\,M_{bh}/R_{bh}^3$, hence smallest holes are densest.
  - Compare density of 10^8Msun BH with Sun (or water) and a giant star (10Rsun).

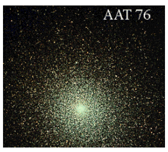
Adiabatic Compression due to growing BH

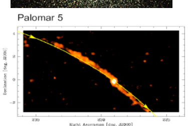
- A star circulating a BH at radius r has
- a velocity $v=(GM_{bh}/r)^{1/2}$,
- an angular momentum $J = r\,v = (GM_{bh}\,r)^{1/2}$,
- As BH grows, Potential and Orbital Energy E changes with time.
- But J conserved (no torque!), still circular!
- So $J_i = (GM_i\,r_i)^{1/2} = J_f = (GM_f\,r_f)^{1/2}$
- Shrink $r_f/r_i = M_i/M_f < 1$, orbit compressed!

Boundary of Star Cluster

- Limitted by tide of Dark-Matter-rich Milky Way

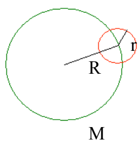






Tidal Stripping

- TIDAL RADIUS: Radius within which a particle is bound to the satellite rather than the host galaxy.
- Consider a satellite (mass m_s) moving in a spherical potential ϕ_g (R) made from a host galaxy (mass M).



If satellite plunges in radially

- the condition for a particle to be bound to the satellite m_s rather than the host galaxy M is:

$$\frac{GM}{(R-r)^2} - \frac{GM}{(R+r)^2} \le \frac{2Gm_s}{r^2}$$

Differential (tidal) force on the particle due to the host galaxy

Force on particle due to satellite

$$\frac{GM}{R^2} U \le \frac{2Gm_s}{r^2}$$

$$\text{If } r << R \text{ then } U = \left(1-\frac{r}{R}\right)^{-2} - \left(1+\frac{r}{R}\right)^{-2} \approx \frac{4r}{R} + \ldots\ldots\ldots$$

$$\rightarrow k\frac{GM}{R^3} \le \frac{Gm_s}{r^3}, k = 2$$

AS4021, Part II 7

Instantaneous Tidal radius

- Generally, $r_t(t) = R(t)\left(\frac{m_s}{kM(R)}\right)^{\frac{1}{3}}$
- fudge factor k varies from 1 to 4 depending on definitions.
- r_t is smallest at pericentre R_p where R is smallest.
- r_t shrinks as a satellite losses mass m.

AS4021, Part II 8

The meaning of tidal radius (k=1)

- Particle Bound to satellite if the mean densities

$$\frac{m_s(r)}{\frac{4}{3}\pi r^3} \ge \frac{M(R)}{\frac{4}{3}\pi R^3}$$

- The less dense part of the satellite is torn out of the system, into tidal tails.

AS4021, Part II 9

Short question

- Recalculate the instantaneous Roche Lobe for satellite on radial orbit, but assume
  Host galaxy potential $\Phi(R) = V_0^2 \ln(R)$
  Satellite self-gravity potential $\varphi(r) = v_0^2 \ln(r)$, where v_0, V_0 are constants.
  - Show $M = V_0^2 R/G$, $m = v_0^2 r/G$,
  - Hence Show $r_t/R =$ cst v_0/V_0, cst $=k^{1/2}$

AS4021, Part II 10

Short questions

- Turn the Sun's velocity direction (keep amplitude) such that the Sun can fall into the BH at Galactic Centre. How accurate must the aiming be in term of angles in arcsec? Find input values from speed of the Sun, BH mass and distances from literature.
- Consider a giant star (of 100solar radii, 1 solar mass) on circular orbit of 0.1pc around the BH, how big is its tidal radius in terms of solar radius? The star will be drawn closer to the BH as it grows. Say BH becomes 1000 as massive as now, what is the new tidal radius in solar radius?

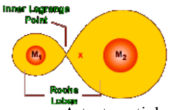
AS4021, Part II 11

Lec 13: rotating potential of satellite-host

- Consider a satellite orbiting a host galaxy
  - Usual energy E and J NOT conserved.
- The frame (x,y,z), in which Φ is static, rotates at angular velocity $\underline{\Omega}_b = \Omega_b\,\underline{e}_z$
- Effective potential & EoM in rotating frame:
  $\ddot{\underline{r}} = -\nabla\Phi_{\rm eff} - 2\left(\underline{\Omega}_b \times \dot{\underline{r}}\right),\ \Phi_{\rm eff} = \Phi - \tfrac{1}{2}\Omega_b{}^2 R^2$
- Prove JACOBI'S ENERGY conserved
  $E_J \equiv E - \underline{\Omega}_b \cdot J = \phi_{eff} + \frac{1}{2}\dot{\underline{r}}^2$

AS4021, Part II 12

Roche Lobe of Satellite

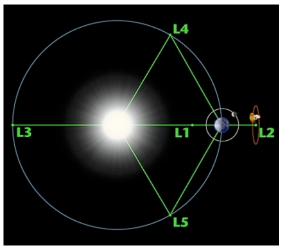


- A test particle with Jakobi energy E_J is bound in a region where $\phi_{eff}(x)<E_J$ since $v^2>0$ always.
- In satellite's orbital plane ($\underline{r}$ perpendicular to $\underline{\Omega}$)

$$\phi_{eff}(\vec{r}) = \phi_g(\vec{R}+\vec{r}) + \phi_s(\vec{r}) - \frac{1}{2}\Omega^2 R^2$$
$$= -\frac{GM}{|\vec{r}+\vec{R}|} - \frac{Gm_s}{|\vec{r}|} - \frac{1}{2}\Omega^2 R^2$$

Lagrange points of satellite

$$\frac{\partial \Phi_{eff}}{\partial x} = 0, \quad \text{and} \quad \frac{\partial \Phi_{eff}}{\partial y} = 0$$



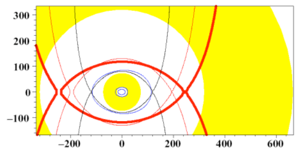
If circular orbit

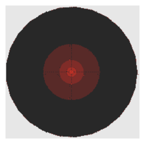
- Rotation angular frequency $\Omega^2 = G(M+m)/R^3$
- L1 point: Saddle point satisfies (after Taylor Expand Φ_{eff} at r=R):

$$r_t = \pm R\left[\frac{m}{M\left(3+\frac{m}{M}\right)}\right]^{\frac{1}{3}} \approx \pm\left(\frac{m}{3M}\right)^{\frac{1}{3}} R$$

- Roche Lobe: equal effective potential contour going through saddle point

Roche Lobe shapes to help Differentiate Newtonian, DM, or MOND





Tidal disruption near giant BH

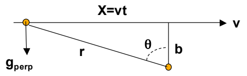
- A giant star has low density than the giant BH, is tidally disrupted first.
- Disruption happens at radius $r_{dis} > R_{bh}$, where $M_{bh}/r_{dis}^3 \sim M_* / R_*^3$
  - Show a giant star is shreded before reaching a million solar mass BH.
- Part of the tidal tail feeds into the BH, part goes out.

What have we learned?

- Criteria to fall into a BH as a whole piece
  - size, loss cone
- Adiabatic contraction
- Tidal disruption criteria
  - Mean density
- Where are we heading?
  - From 2-body to N-body system

Lec 14: Encounter a star occasionally

- Orbit deflected
- evaluate deflection of a particle when encountering a star of mass m at distance b:



$$g_\perp = \frac{Gm}{r^2}\cos\theta = \frac{Gm\,b}{(x^2+b^2)^{3/2}} \approx \frac{Gm}{b^2}\left[1+\left(\frac{vt}{b}\right)^2\right]^{-3/2}$$

Stellar Velocity Change Δv_{perp}

- sum up the impulses dt g_{perp}
  - use s = vt / b

$$\Delta v_\perp = \int g_\perp dt = \frac{Gm}{bv}\int_{-\infty}^{\infty}(1+s^2)^{3/2}ds = \frac{2Gm}{bv}$$

$$\Delta v_\perp = g_\perp \Delta t = \frac{Gm}{b^2}\times\frac{2b}{v}$$

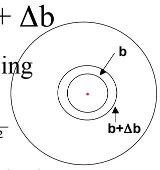
- Or using impulse approximation:
  - where g_{perp} is the force at closest approach and
  - the duration of the interaction can be estimated as : $\Delta t = 2\,b\,/\,v$

Crossing a system of N stars plus Dark Matter elementary particles

- let system diameter be: 2R
- Argue Crossing time t_{cross} =2R/v
- Star number density per area ~ N/($R^2\pi$)
- Total mass M =N*m* + $N_{dm}m_{dm}$ $^>$ N*m*
- Typically
  - m_{dm}~ 1Gev << m* = m
  - N_{dm} > 10^{20} > N* = N

Number of encounters with impact parameter b - b + Δb

- # of stars on the way per crossing



$$\delta N = \left[\pi(b+\Delta b)^2 - \pi b^2\right]\frac{N}{\pi R^2}$$

$$\cong \frac{2\pi b\Delta b N}{\pi R^2} = \frac{2N}{R^2}b\Delta b$$

$$\langle \Delta v_\perp \rangle = 0$$

- each encounter is randomly oriented
- sum is zero:

Sum up the heating in kinetic energy

- sum over gain in $(\Delta v_{perp}{}^2)/2$ in one-crossing

$$\frac{1}{2}\langle\Delta v_\perp{}^2\rangle = \int\left[\frac{2Gm}{bv}\right]^2\frac{N}{R^2}b\Delta b = \int_{b\min}^{b\max}\frac{db}{b}N\left[\frac{2Gm}{Rv}\right]^2$$

$$\langle\Delta v_\perp{}^2\rangle/v^2 \approx 8N\left[\frac{m}{M}\right]^2\ln\Lambda,\ \text{where}\ \Lambda = \frac{b\max}{b\min}\sim N$$

- consider encounters over all b
  - b< bmax ~ R ~ GM/v^2 [M= total mass of system]
  - b> bmin ~ R/N

Relaxation time

- Orbit Relaxed after n_{relax} times across the system so that orbit deflected by $\Delta v^2/v^2$ ~1

$$n_{relax} \approx \frac{v^2}{\Delta v^2} \approx \frac{N'}{8\ln\Lambda} \qquad \text{where } N' = (M/m)^2/N \geq N$$

- thus the relaxation time is:

$$t_{relax} = n_{relax}t_{cross} \approx \frac{N'}{8\ln\Lambda}t_{cross}$$

- Argue two-body scattering between star-star, star-DM, lump-star, lump-DM are significant, but not between 1Gev particles.

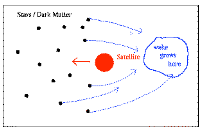
How long does it take for real systems to relax?

- globular cluster, $N=10^5$, $R=10$ pc
  - $t_{cross} \sim 2\ R / v \sim 10^5$ years
  - $t_{relax} \sim 10^8$ years << age of cluster: relaxed
- galaxy, $N=10^{11}$, $R=15$ kpc
  - $t_{cross} \sim 10^8$ years
  - $t_{relax} \sim 10^{15}$ years >> age of galaxy: collisionless
- cluster of galaxies: $t_{relax} \sim$ age

Self-heating/Expansion/Segregation of an isolated star cluster: Relax!

- Core of the cluster contracts, form a tight binary with very negative energy
- Outer envelope of cluster receives energy, becomes bigger and bigger.
  - Size increases by order 1/N per crossing time.
  - Argue a typical globular cluster has size-doubled
- Low-mass stars segregate and gradually diffuse out/escape

Lec 15: Dynamical Friction



- As the satellite moves through a sea of background particles, (e.g. stars and dark matter in the parent galaxy) the satellites gravity alters the trajectory of the background stars, building up a slight density enhancement of stars behind the satellite
- The gravity from the wake pulls backwards on the satellites motion, slowing it down a little

- This effect is referred to as "dynamical friction" because
  - it acts like a frictional or viscous force,
  - but it's pure gravity.
- It creates density wakes at low speed,
  - & cone-shaped wakes if satellite travels with high speed.

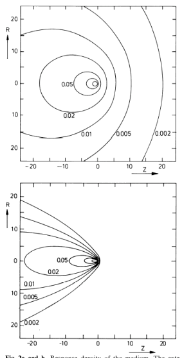


Fig. 2a and b. Response density of the medium. The extended object moves in the positive z-direction. Contours represent overdensities of 0.002, 0.005, 0.01, 0.02, 0.05, 0.1, and 0.2, using $\varrho_0=1$. Parameters are for **a** $v_0=2$, $\sigma_0=2$, and for **b** $v_0=6$ and $\sigma_0=2$

Chandrasekhar Dynamical Friction Formula

- The dynamical friction acting on a satellite of mass M moving at v_s kms^{-1} in a sea of particles of density m*n(r) with Gaussian velocity distribution

$$f(\vec{r},\vec{v}) = f(r,0)\exp\left(-\frac{v_s^2}{2\sigma^2}\right), f(r,0) = \frac{n(r)m}{\left(\sqrt{2\pi\sigma}\right)^3}$$

- Only stars moving slower than M contribute to the force.

$$\frac{dv_M}{dt} = -16\pi^2 \ln\Lambda G^2 (M+m) \frac{\int_0^{v_M} f(v_m) v_m^2 dv_m}{v_M^3} V_M$$

Dependence on satellite speed

- For a sufficiently large v_M, the integral converges to a definite limit and the frictional force therefore falls like v_M^{-2}.
- For sufficiently small v_M we may replace $f(v_M)$ by $f(0)$, hence force goes up with v_M:

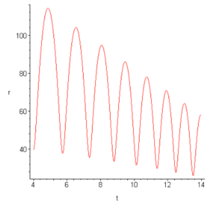
$$\frac{dv_M}{dt} \approx -\frac{16\pi^2}{3} \ln\Lambda G^2 f(0)(M+m) v_M = -\frac{v_M}{t_{fric}}$$

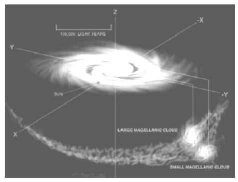
  - This defines a typical friction timescale t_{fric}

Depends on M, $n^*(r)m^*$ & $n_{dm}(r)m_{dm}$

- More massive satellites feel a greater friction
  - since they can alter trajectories more and build up a more massive wake behind them.
- Dynamical friction is stronger in higher density regions
  - since there are more stars to contribute to the wake so the wake is more massive.
- Note: both stars (m*~Msun) and dark matter particles (m_{dm}~1Gev) contribute to dynamical friction.

Friction & tide: effects on satellite orbit

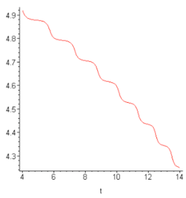




- The drag force dissipates orbital energy E(t) and J(t)
  - The decay is faster at pericentre
  - staircase-like decline of E(t), J(t).
- As the satellite moves inward the tidal becomes greater
  - so the tidal radius decreases and the mass m(t) will decay.

Orbital decay of Large Magellanic Cloud: a proof of dark matter?

- Dynamical friction to drag LMC's orbit at R=50-100 kpc:
  - density of stars from Milky Way at 50 kpc very low
    - No drag from ordinary stars
  - dark matter density is high at 50 kpc
    - Drag can only come from dark matter particles in Milky Way
- Energy (from future velocity data from GAIA) difference earlier/later debris on the stream may reveal evidences for orbital decay



Summary

- Relaxation is a measure of granularity in potential of N-particles of different masses
  - Relaxation cause energy diffusion from core to envelope of a system,
  - expansion of the system,
  - evaporation (~escape) of stars
- Massive lumps leaves wakes, transport energy/momentum to background.
  - Cause orbit decay,
  - galaxies merge

Tutorial session