Gravitational Dynamics: An Introduction

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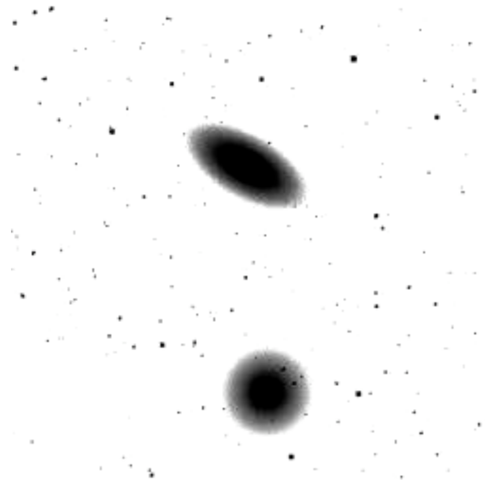
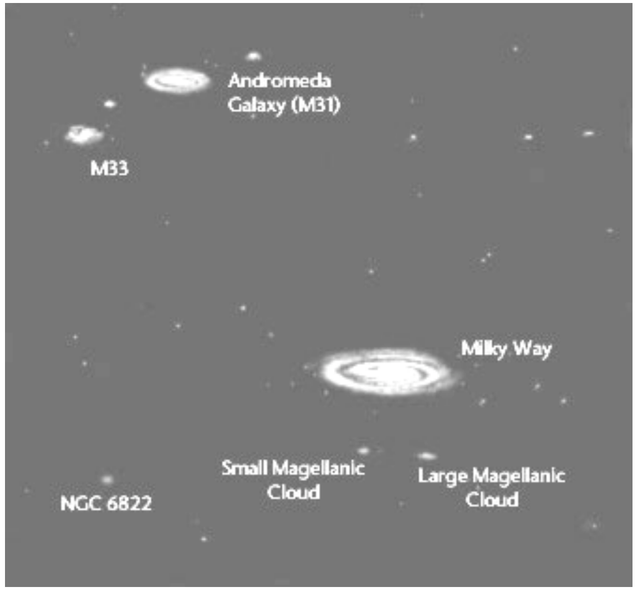
C1.1.1 Our Galaxy and Neighbours



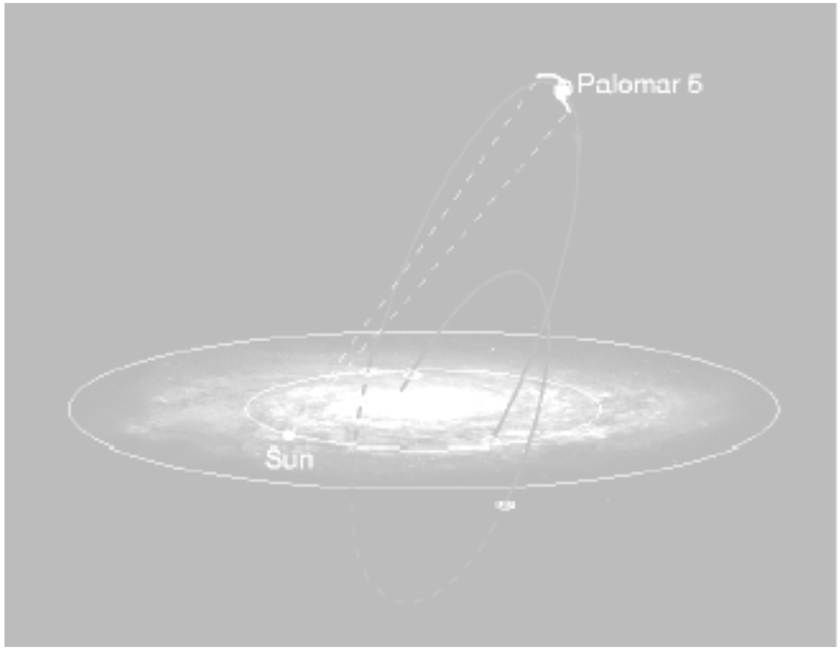


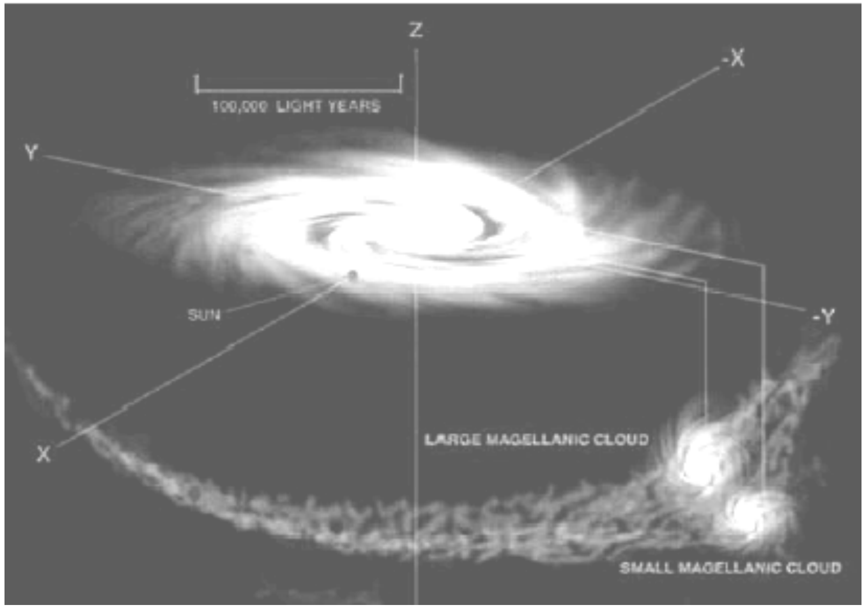
- How structure in universe form/evolve?
- Galaxy Dynamics Link together early universe & future.

Our Neighbours

- M31 (now at 500 kpc) separated from MW a Hubble time ago
- Large Magellanic Cloud has circulated our Galaxy for about 5 times at 50 kpc
  - **argue** both neighbours move with a typical 100-200km/s velocity relative to us.

Outer Satellites on weak-g orbits around Milky Way





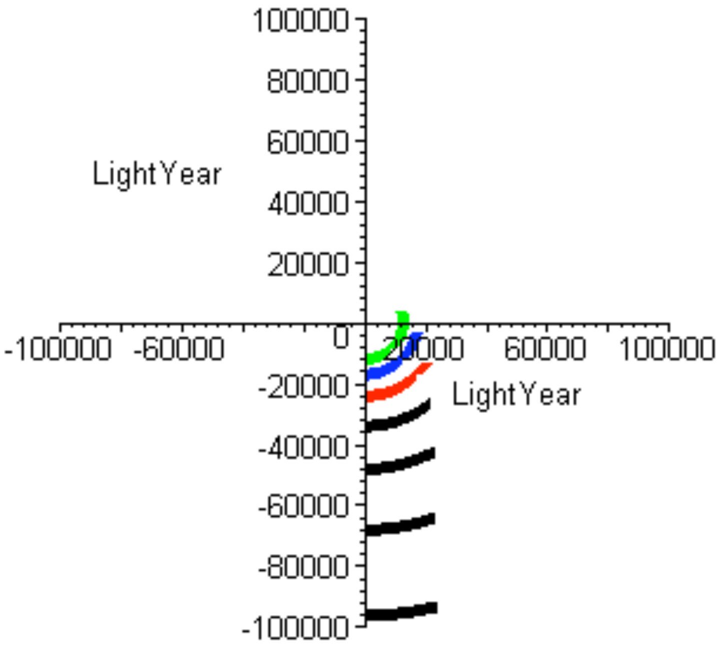
R>10kpc: Magellanic/Sgr/Canis streams
R>50kpc: Draco/Ursa/Sextans/Fornax…

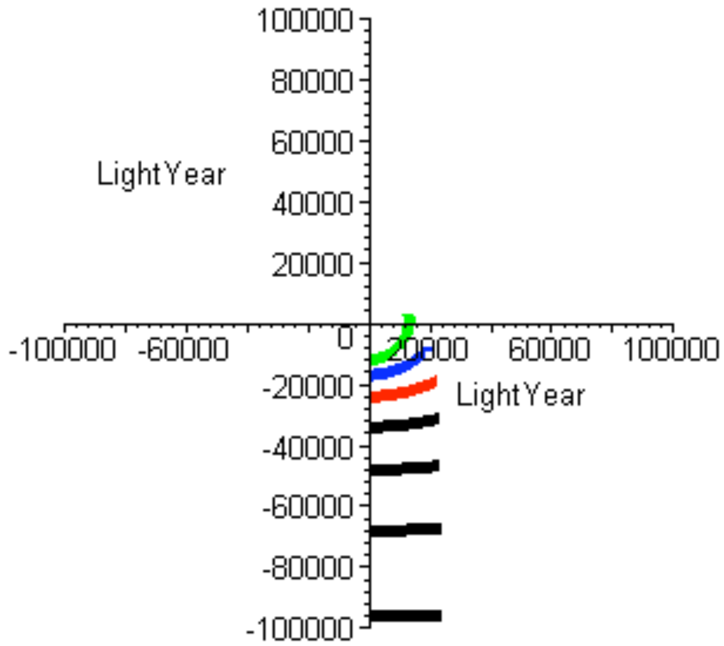
~ 50 globulars on weak-g (R<150 kpc)
~100 globulars on strong-g (R< 10 kpc)

C1.1.2 Milky Way as Gravity Lab

- Sun has circulated the galaxy for 30 times
  - velocity vector changes direction +/- 200km/s twice each circle (R = 8 kpc)
  - **Argue** that the MW is a nano-earth-gravity Lab
  - **Argue** that the gravity due to 10^{10} stars only within 8 kpc is barely enough. Might need to add Dark Matter.

Sun escapes unless our Galaxy has Dark Matter





C1.1.3 Dynamics as a tool

- Infer additional/dark matter
  - – E.g., Weakly Interacting Massive Particles
    - proton mass, but much less interactive
    - Suggested by Super-Symmetry, but undetected
  - – A $billion$ industry to find them.
    - What if they don’t exist?

. . .

- Test the law of gravity:
  - valid in nano-gravity regime?
  - Uncertain outside solar system:
    - GM/r^2 or cst/r ?

Outer solar system

- The Pioneer experiences an anomalous non-Keplerian acceleration of 10^{-8} cm s^{-2}
  - What is the expected acceleration at 10 AU?
  - What could cause the anomaly?

Gravitational Dynamics can be applied to:

- Two body systems:binary stars
- Planetary Systems, Solar system
- Stellar Clusters:open & globular
- Galactic Structure:nuclei/bulge/disk/halo
- Clusters of Galaxies
- The universe:large scale structure

Topics

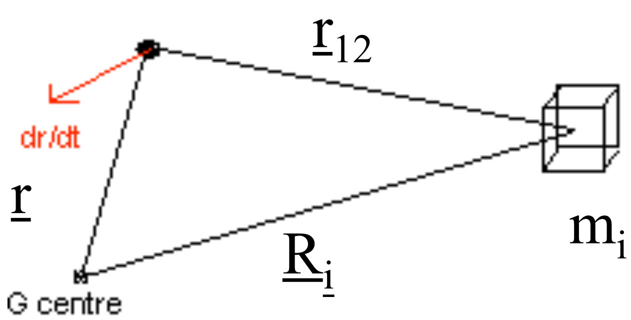
- Phase Space Fluid f(x,v)
  - Eq^n of motion
  - Poisson's equation
- Stellar Orbits
  - Integrals of motion (E,J)
  - Jeans Theorem
- Spherical Equilibrium
  - Virial Theorem
  - Jeans Equation
- Interacting Systems
  - Tides→Satellites→Streams
  - Relaxation→collisions
- MOND

C2.1 How to model motions of 10^{10}stars in a galaxy?

- Direct N-body approach (as in simulations)
  - At time t particles have $(m_i,x_i,y_i,z_i,vx_i,vy_i,vz_i)$, $i=1,2,...,N$ (feasible for $N<<10^6$).
- Statistical or fluid approach (N very large)
  - At time t particles have a spatial density distribution $n(x,y,z)*m$, e.g., uniform,
  - at each point have a velocity distribution $G(vx,vy,vz)$, e.g., a 3D Gaussian.

C2.2 N-body Potential and Force

- In N-body system with mass $m_1 \ldots m_N$, the gravitational acceleration $\underline{g}(r)$ and potential $\varphi(r)$ at position $\underline{r}$ is given by:



$$\vec{F} = m\vec{g}(r) = -\sum_{i=1}^{N} \frac{G \cdot m \cdot m_i \cdot \hat{r}_{12}}{\left|\vec{r} - \vec{R}_i\right|^2} = -m\vec{\nabla}_r \phi$$

$$\Phi = m\phi(r) = -\sum_{i=1}^{N} \frac{G \cdot m \cdot m_i}{\left|\vec{r} - \vec{R}_i\right|}$$

Example: Force field of two-body system in Cartesian coordinates

$$\phi(\vec{r}) = -\sum_{i=1}^{2} \frac{G \cdot m_i}{\left|\vec{r} - \vec{R}_i\right|}, \text{ where } \vec{R}_i = (0,0,-i) * a, m_i = m_\circ$$

Sketch the configuration, sketch equal potential contours

$$\phi(x, y, z) = ?$$

$$\vec{g}(\vec{r}) = (g_x, g_y, g_z) = -\nabla\phi(\vec{r}) = \left(-\frac{\partial\phi}{\partial x}, -\frac{\partial\phi}{\partial y}, -\frac{\partial\phi}{\partial z}\right)$$

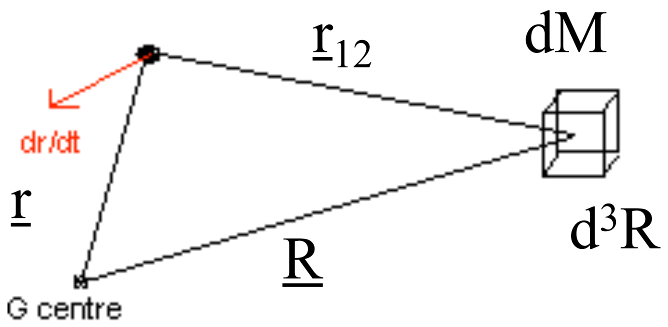
$$\left\|\vec{g}(\vec{r})\right\| = \sqrt{(g_x^2 + g_y^2 + g_z^2)} = ?$$

sketch field lines. at what positions is force = 0?

C2.3 A fluid element: Potential & Gravity

- For large N or a continuous fluid, the gravity dg and potential dφ due to a small mass element dM is calculated by replacing m_i with dM:

$$d\vec{g} = -\frac{G \cdot dM \cdot \hat{r}_{12}}{\left|\vec{r} - \vec{R}_i\right|^2}$$



$$d\phi = -\frac{G \cdot dM}{\left|\vec{r} - \vec{R}\right|}$$

Lec 2: Why Potential $\varphi(r)$?

- Potential per unit mass $\varphi(r)$ is scalar,
  - function of $\underline{r}$ only,
  - Related to but easier to work with than force (vector, 3 components)
  - Simply relates to orbital energy $E= \varphi(r) + \frac{1}{2} v^2$

C2.4 Poisson's Equation

- PE relates the potential to the density of matter generating the potential by:

$$\nabla \cdot \nabla \phi = -\vec{\nabla} \cdot \vec{g} = 4\pi G \rho(r)$$

- [BT2.1]

C2.5 Eq. of Motion in N-body

- Newton's law: a point mass m at position $\underline{r}$ moving with a velocity $d\underline{r}/dt$ with Potential energy $\Phi(\underline{r}) = m\varphi(r)$ experiences a Force $F = mg$, accelerates with following Eq. of Motion:

$$\frac{d}{dt}\left[\frac{d\vec{r}(t)}{dt}\right] = \frac{\vec{F}}{m} = \frac{-\vec{\nabla}_{\vec{r}}\Phi(r)}{m}$$

Example 1: trajectories when G=0

- Solve Poisson's Eq. with G=0 ➔
  - **F**=0, ➔ $\Phi(\underline{r})$=cst, ➔
- Solve EoM for particle i initially at ($\mathbf{X}_{0,i}$, $\mathbf{V}_{0,i}$)
  - $d\mathbf{V}_i/dt = \mathbf{F}_i/m_i = 0$ ➔ $\mathbf{V}_i = cst = \mathbf{V}_{0,i}$
  - $d\mathbf{X}_i/dt = \mathbf{V}_i = \mathbf{V}_{i,0}$ ➔ $\mathbf{X}_i(t) = \mathbf{V}_{0,i}\, t + \mathbf{X}_{0,i}$,
  - where **X**, **V** are vectors,
  - ➔straight line trajectories
- E.g., photons in universe go straight
  - occasionally deflected by electrons,
  - Or bent by gravitational lenses

What have we learned?

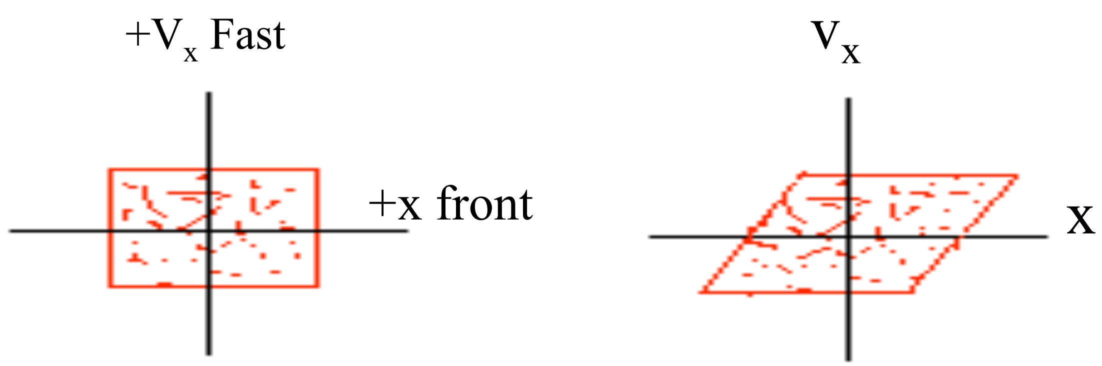
- Implications on gravity law and DM.
- Poisson’s eq. and how to calculate gravity
- Equation of motion

How N-body system evolves

- Start with initial positions and velocities of all N particles.
- Calculate the mutual gravity on each particle
  - Update velocity of each particle for a small time step dt with EoM
  - Update position of each particle for a small time step dt
- Repeat previous for next time step.
- ➔ N-body system fully described

C2.6 Phase Space of Galactic Skiers

- N_{skiers} identical particles moving in a small bundle in phase space (Vol $= \Delta_x \Delta_v$),
- phase space deforms but maintains its area.



- Gap widens between faster & slower skiers
  - but the phase volume & No. of skiers are constants.

“Liouvilles Theorem on the piste”

- Phase space density of a group of skiers is const.

  $f = m \; N_{skiers} / \Delta x \, \Delta v_x = \text{const}$

  Where m is mass of each skier,

  [BT4.1]

C2.7 density of phase space fluid: Analogy with air molecules

- air with uniform density n=10^{23} cm^{-3}

  Gaussian velocity rms velocity σ =0.3km/s in x,y,z directions:

$$f(x, v) = \frac{m \times n_o \exp\left(-\frac{v_x^2 + v_y^2 + v_x^2}{2\sigma^2}\right)}{(\sqrt{2\pi\sigma})^3}$$

- Estimate f(0,0,0,0,0,0)/m in pc^{-3} $(km/s)^{-3}$

Lec 3 (Tuesday)

C2.8 Phase Space Distribution Function (DF)

PHASE SPACE DENSITY: No. of sun-like stars per unit volume per velocity volume f(**x**,**v**)

$$f(x,v) = \frac{dN \times m_{sun}}{dx^3 dv^3} = \frac{\text{number of suns} \times m_{sun}}{\text{space volume} \times \text{velocity volume}}$$

$$\square \; \frac{1 \times m_{sun}}{pc^3 \times (100 kms^{-1})^3}$$

C2.9 add up stars: integrate over phase space

- star mass density: integrate velocity volume

$$\rho(\vec{x}) = m_{sun} \times n(\vec{x}) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(\vec{x},\vec{v})d\vec{v}_x d\vec{v}_y d\vec{v}_z$$

- The total mass : integrate over phase space

$$M_{total} = \int \rho(x) d^3x = \int f(\vec{x},\vec{v}) d^3\vec{v} d^3\vec{x}$$

- define spatial density of stars n(x)

$$n \equiv \int f d^3 v$$

- and the mean stellar velocity v(x)

$$n\bar{v}_i \equiv \text{flux in i-direction} = \int f v_i d^3 v$$

- E.g., Conservation of flux (without proof)

$$\frac{\partial n}{\partial t} + \frac{\partial \left(n\bar{v}_1\right)}{\partial x_1} + \frac{\partial \left(n\bar{v}_2\right)}{\partial x_2} + \frac{\partial \left(n\bar{v}_3\right)}{\partial x_3} = 0$$

C3.0 Star clusters differ from air:

- Stars collide far less frequently
  - size of stars<<distance between them
  - Velocity distribution not isotropic
- Inhomogeneous density $\rho(r)$ in a Grav. Potential $\varphi(\mathbf{r})$

Example 2: A 4-body problem

- Four point masses with G m = 1 at rest (x,y,z)=(0,1,0),(0,-1,0),(-1,0,0),(1,0,0). Show the initial total energy

  $Einit = 4 * (½ + 2^{-1/2} + 2^{-1/2}) /2 = 3.8$

- Integrate EoM by brutal force for one time step =1 to find the positions/velocities at time t=1.
  - Use $V=V_0 + g\,t = g = (u, u, 0)$; $u = 2^{1/2}/4 + 2^{1/2}/4 + ¼ = 0.95$
  - Use $x= x_0 + V_0\,t = x_0 = (0, 1, 0)$.

- How much does the new total energy differ from initial?

  $E - Einit = ½ (u^2 +u^2) * 4 = 2 u^2 = 1.8$

Often-made Mistakes

- Specific energy or specific force confused with the usual energy or force
- Double-counting potential energy between any pair of mass elements, kinetic energy with v^2

- Velocity vector V confused with speed,
- $1/|r|$ confused with $1/|x|+1/|y|+1/|z|$

What have we learned?

Potential to Gravity $$g = -\nabla\phi$$

Potential to density $$\rho = \frac{1}{4\pi G}\nabla^2\phi$$

Density to potential $$\phi(\vec{r}) = -\int \frac{G\rho d^3\vec{r}'}{\left|\vec{r} - \vec{r}'\right|}$$

Motion to gravity $$g = dv/dt$$

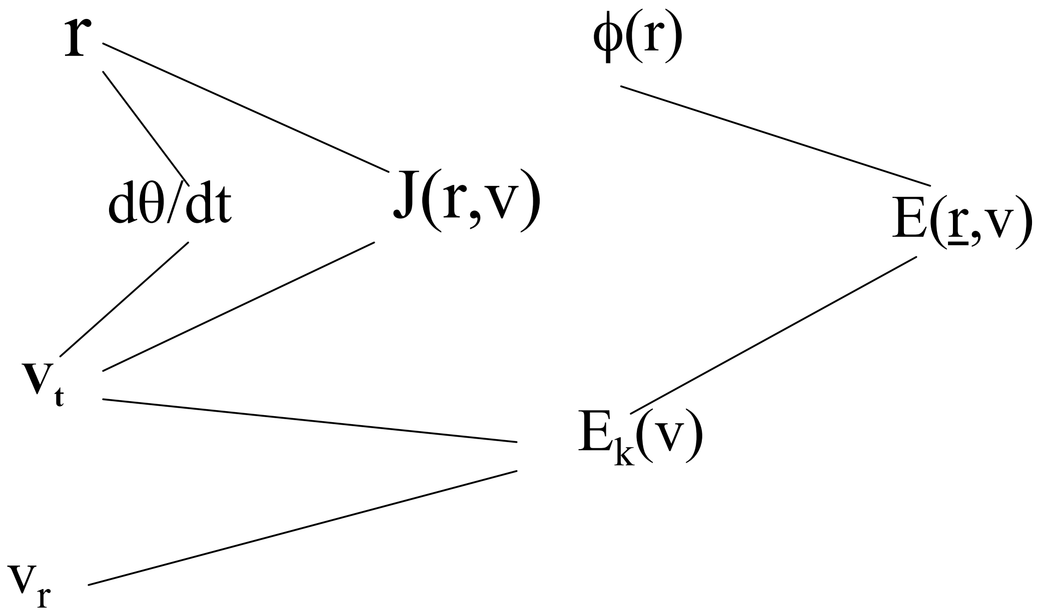
Concepts

- Phase space density
  - incompressible
  - Dimension Mass/[Length^3 Velocity^3]
  - **Show** a pair of non-relativistic Fermionic particle occupy minimal phase space $(x*v)^3 > (h/m)^3$, hence has a maximum phase density $=2m\,(h/m)^{-3}$

Where are we heading to?
Lec 4, Friday 22 Feb

- potential and eqs. of motion
  - in general geometry
  - Axisymmetric
  - spherical

Link phase space quantities



C 3.1: Laplacian in various coordinates

Cartesians :

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Cylindrical :

$$\nabla^2 = \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial}{\partial R}\right) + \frac{1}{R^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}$$

Spherical :

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2}{\partial \phi^2}$$

Example 3: Energy is conserved in STATIC potential

- The orbital energy of a star is given by:

$$E = \frac{1}{2}v^2 + \phi(\vec{r}, t)$$

$$\frac{dE}{dt} = \underline{\underline{\vec{v}\frac{d\vec{v}}{dt} + \frac{d\vec{r}}{dt}\nabla\phi}} + \underline{\underline{\frac{\partial\phi}{\partial t}}} = 0 + \underline{\underline{\frac{\partial\phi}{\partial t}}}$$

0 since $\frac{d\vec{v}}{dt} = -\nabla\phi$

and $\frac{d\vec{r}}{dt} = \vec{v}$

0 for static potential.

So orbital Energy is Conserved dE/dt=0 only in "time-independent" potential.

Example 4: Static Axisymmetric density → Static Axisymmetric potential

- We employ a cylindrical coordinate system (R, θ,z) e.g., centred on the galaxy and align the z axis with the galaxy axis of symmetry.
- Here the potential is of the form φ(R,z).
- Density and Potential are Static and Axisymmetric
  - independent of time and azimuthal angle

$$\phi(R,z) \Rightarrow \rho(R,z) = \frac{1}{4\pi G}\left[R\frac{\partial}{\partial R}\left(R\frac{\partial\phi}{\partial R}\right) + \frac{\partial^2\phi}{\partial z^2}\right]$$

$$g_r = -\frac{\partial\phi}{\partial R} \qquad g_z = -\frac{\partial\phi}{\partial z}$$

C3.2: Orbits in an axisymmetric potential

- Let the potential which we assume to be symmetric about the plane z=0, be $\phi(R,z)$.
- The general equation of motion of the star is

$$\frac{d^2\vec{r}}{dt^2} = -\nabla\phi(R,z) \qquad \text{Eq. of Motion}$$

- Eqs. of motion in cylindrical coordinates

$$\ddot{z} = -\frac{\partial\phi}{\partial z}, \quad \ddot{R} - R\dot{\theta}^2 = -\frac{\partial\phi}{\partial R}, \; 2\dot{R}\dot{\theta} + R\ddot{\theta} = \frac{d}{R\,dt}(R^2\dot{\theta}) = -\frac{\partial\phi}{R\partial\theta} = 0$$

Conservation of angular momentum z-component Jz if axisymmetric

$$J_Z = R^2\dot{\theta} \Rightarrow \frac{d}{dt}Jz = \frac{d}{dt}(R^2\dot{\theta}) = 0$$

- The component of angular momentum about the z-axis is conserved.

- If $\phi(R,z)$ has no dependence on θ then the azimuthal angular momentum is conserved
  - or because z-component of the torque $\underline{r} \times \underline{F}=0$. (Show it)

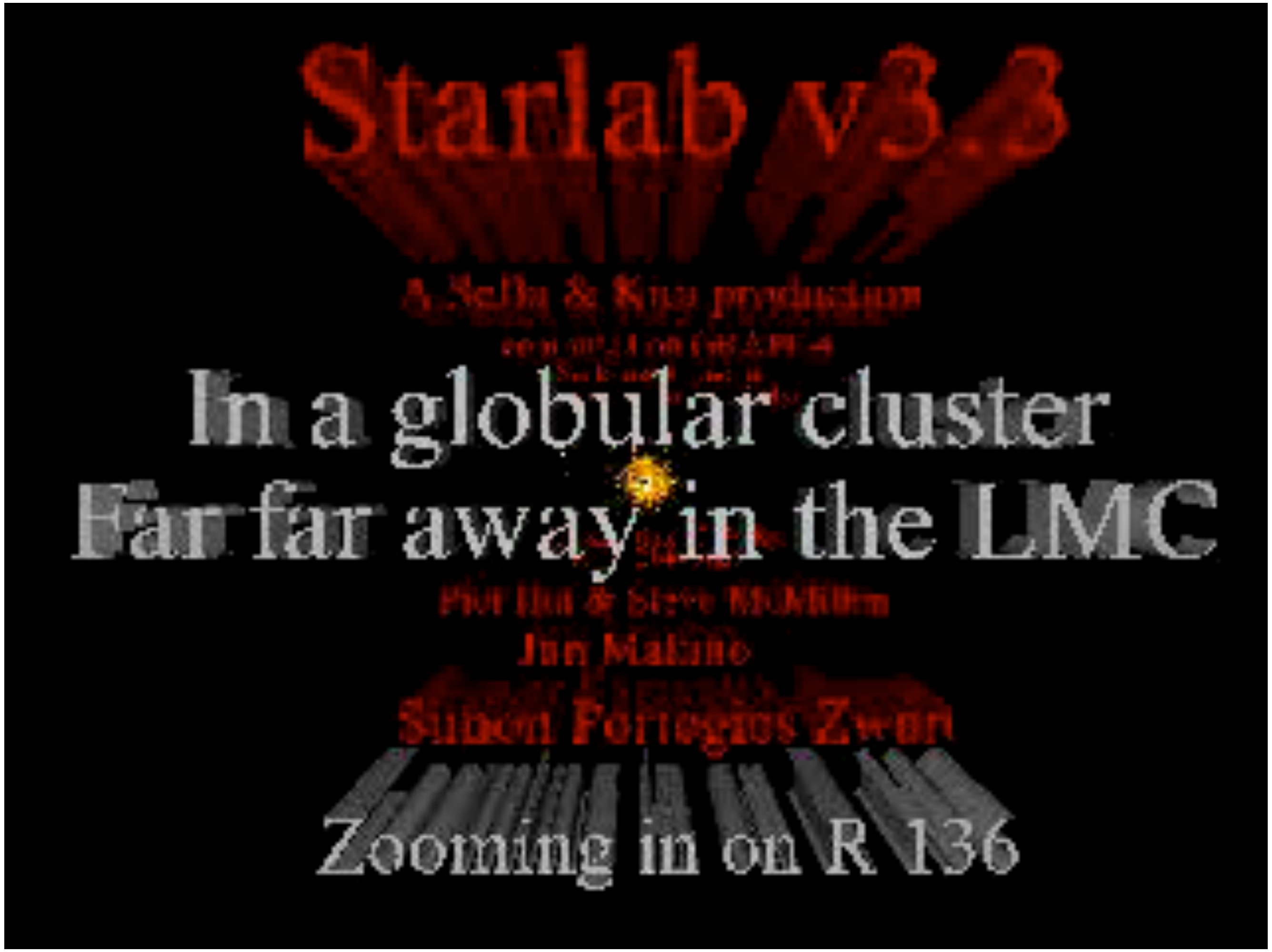
C4.1: Spherical Static System

- Density, potential function of radius |r| only
- Conservation of
  - energy E,
  - angular momentum J (all 3-components)
  - Argue that a star moves orbit which confined to a plane perpendicular to J vector.

C 4.1.0: Spherical Cow Theorem

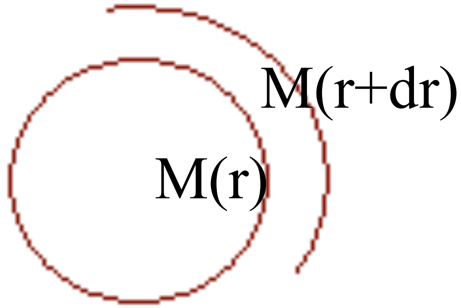
- Most astronomical objects can be approximated as spherical.
- Anyway non-spherical systems are too difficult to model, almost all models are spherical.

Globular: A nearly spherical static system



C4.2: From Spherical Density to Mass

$$M(R + dr) = M(R) + dM$$



$$dM = \rho(r) d\left(\frac{4}{3}\pi r^3\right) = 4\pi r^2 \rho(r) dr$$

$$\rho(r) = \frac{dM}{d\left(\frac{4}{3}\pi r^3\right)} = \frac{dM}{4\pi r^2 dr}$$

$$M(R) = \int \rho d\left(\frac{4}{3}\pi r^3\right)$$

C4.3: Theorems on Spherical Systems

- NEWTONS 1st THEOREM: A body that is inside a spherical shell of matter experiences no net gravitational force from that shell
- NEWTONS 2nd THEOREM: The gravitational force on a body that lies outside a closed spherical shell of matter is the same as it would be if all the matter were concentrated at its centre. [BT 2.1]

C4.4: Poisson's eq. in Spherical systems

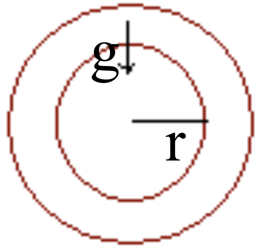
- Poisson's eq. in a spherical potential with no θ or Φ dependences is:

$$\nabla^2 \varphi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \varphi}{\partial r}\right) = 4\pi G \rho(r)$$

- BT2.1.2

Example 5: Interpretation of Poissons Equation

- Consider a spherical distribution of mass of density ρ(r).



$$g = -\frac{GM(r)}{r^2}$$

$$\phi = \int_r^\infty g(r)dr \qquad \text{since } \phi = 0 \text{ at } \infty \text{ and is} < 0 \text{ at r}$$

$$= -\int_r^\infty \frac{GM(r)}{r^2}dr$$

$$\text{Mass Enclosed} = \int_r^\infty 4\pi r^2 \rho(r)dr$$

- Take d/dr and multiply r² →

$$r^2 \frac{d\phi}{dr} = -gr^2 = GM(r) = \left(G\int 4\pi r^2 \rho(r) dr \right)$$

- Take d/dr and divide r²→

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \varphi}{\partial r} \right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(-r^2 g \right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(GM \right) = 4\pi G\rho(r)$$

$$\rightarrow \nabla^2 \phi = -\vec{\nabla}.g = 4\pi G\rho$$

C4.5: Escape Velocity

- ESCAPE VELOCITY= velocity required in order for an object to escape from a gravitational potential well and arrive at ∞ with zero KE.

=0 often

$$\phi(r) = \phi(\infty) - \frac{1}{2}v_{esc}^2$$

$$\rightarrow v_{esc}(r) = \sqrt{2\phi(\infty) - 2\phi(r)}$$

Example 6: Plummer Model for star cluster

- A spherically symmetric potential of the form:

$$\phi = -\frac{GM}{\sqrt{r^2 + a^2}}$$

e.g., for a globular cluster a=1pc, M=10^5 Sun Mass show Vesc(0)=30km/s

- Show corresponding to a density (use Poisson's eq):

$$\rho = \frac{3M}{4\pi a^3}\left(1 + \frac{r^2}{a^2}\right)^{-\frac{5}{2}}$$

What have we learned?

- Conditions for conservation of orbital energy, angular momentum of a test particle
- Meaning of escape velocity
- How Poisson’s equation simplifies in cylindrical and spherical symmetries