GENERAL RELATIVITY

Additional problems - Solutions

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1. **According to the notation used in the lecture, the following solution is based on a metric signature** $(-,-,-,+)$ **and a definition of the Ricci tensor** $R_{\mu\nu}$ **given by**

$$R_{\mu\nu} = g^{\lambda\kappa} R_{\kappa\mu\nu\lambda} = R^{\lambda}{}_{\mu\nu\lambda}. \tag{1}$$

Using this particular convention, Einstein's equations take the form

$$G^{\mu\nu} = -\frac{8\pi G}{c^4} T^{\mu\nu} - \Lambda g^{\mu\nu}. \tag{2}$$

We start from a FLRW metric which is given as

$$ds^2 = c^2 dt^2 - a(t)^2 \left(\frac{dr^2}{1-kr^2} + r^2 (d\theta^2 + \sin^2\theta d\phi^2) \right). \tag{3}$$

Using the fact that a perfect fluid is described by

$$T^{\mu\nu} = \left(\rho + \frac{P}{c^2}\right) u^\mu u^\nu - P g^{\mu\nu} \tag{4}$$

and assuming that the fluid is at rest,

$$u^4 = c, \quad u^i = 0, \qquad i = 1..3, \tag{5}$$

we obtain

$$\begin{aligned} \frac{8\pi G}{c^4} T^{44} + \Lambda g^{44} &= \frac{8\pi G}{c^2} \rho + \Lambda, \\ \frac{8\pi G}{c^4} T^{ij} + \Lambda g^{ij} &= -\frac{8\pi G}{c^4} P g^{ij} + \Lambda g^{ij}, \qquad i,j = 1..3. \end{aligned} \tag{6}$$

Now let us substitute the FLRW metric into Einstein's equations, i.e. we calculate the corresponding Einstein tensor $G^{\mu\nu} = R^{\mu\nu} - Rg^{\mu\nu}/2$ **(dot denotes derivatives w.r.t.** t**):**

$$\begin{aligned} G^{44} &= -\frac{3}{c^2}\left(\frac{\dot a}{a}\right)^2 - \frac{3k}{a^2}, \\ G^{ij} &= -\left(\frac{2\ddot a}{c^2 a} + \frac{\dot a^2}{c^2 a^2} + \frac{k}{a^2}\right) g^{ij}. \end{aligned} \tag{7}$$

Combining equations (6) **and** (7)**, we finally arrive at Friedmann's equations:**

$$\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2} + \frac{\Lambda c^2}{3}, \qquad \frac{\ddot a}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{P}{c^2}\right) + \frac{\Lambda c^2}{3}. \tag{8}$$

Clearly, the above equation implies $\Lambda = 4\pi G\rho/c^2$ for a static and pressureless universe ($\ddot{a} = \dot{a} = P = 0$), and we finally end up with

$$-\frac{G^{44}}{3} = \frac{k}{a^2} = \Lambda, \qquad -G^{ij} = \frac{k}{a^2}g^{ij} = \Lambda g^{ij}, \qquad i,j = 1..3. \tag{9}$$

Since $g_{\mu\nu}$ is diagonal, g^{ij} and g_{ij} are diagonal, and only the components G^{ii} and G^{44} are non-zero. Furthermore, if $\rho > 0$, the first equation in (8) yields $k > 0$ for our particular choice of Λ. In this case, the static solution has positive spatial curvature and corresponds to a closed universe. If the matter density is zero everywhere, the solution corresponds to an empty universe without expansion, cosmological constant and spatial curvature ($k = \Lambda = 0$), hence to a flat Minkowski spacetime.

2. Obviously, we can use the Schwarzschild metric, which is a vacuum solution to Einstein's equations, to describe the exterior of star-like objects. Consider the following form of Einstein's equations (cosmological constant $\Lambda = 0$)

$$R_{\mu\nu} = -\frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right). \tag{10}$$

Since $T_{\mu\nu} = T = 0$ outside the star, we immediately find $R_{\mu\nu}$=0. Therefore, also its contraction $R = g^{\mu\nu}R_{\mu\nu}$ is zero.

However, both $R_{\mu\nu}$ and R do not contain the full information about the spacetime geometry ($R_{\mu\nu}$ becomes incomplete for $d = 4$ dimensions or higher, R already for $d = 3$). The tensor which fully characterizes the curvature of spacetime is the Riemannian curvature tensor $R_{\mu\nu\sigma\kappa}$. An elementary calculation reveals that there are non-vanishing components of this tensor, and thus the Schwarzschild spacetime has to be curved.

3. The Schwarzschild metric reads as follows (expressed in Schwarzschild coordinates):

$$ds^2 = \left(1 - \frac{r_s}{r}\right)c^2dt^2 - \left(1 - \frac{r_s}{r}\right)^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{11}$$

where $r_s = 2GM/c^2$ is the Schwarzschild radius. Assuming that the astronauts are (and approximately stay) at the same place, we only have to consider the time dilatation between the different reference frames. To relate the proper time τ (different for each astronaut) to the receiver's time on earth, we use the general expression

$$cd\tau = \sqrt{g_{\mu\nu}dx^\mu dx^\nu} = \sqrt{\left(1 - \frac{r_s}{r}\right)c^2dt^2 - \left(1 - \frac{r_s}{r}\right)^{-1}dr^2 - r^2d\phi^2}, \tag{12}$$

where we have set $\theta = \pi/2$ without loss of generality. First of all, we note that there are two different effects causing time dilatation between the reference frames: the geometry of spacetime, i.e. gravitation, and the astronauts' motion within that spacetime, i.e. dynamics. Both of them will contribute, and since the underlying Schwarzschild geometry is the same for every astronaut, the signal sent out by S will be the most frequent one (F and C are additionally in motion, which gives a larger time delay). This is how a simple answer could look like.

However, it is also possible to calculate the time delays for each case, which will also enable us to explore the difference between astronaut F and C. For astronaut S, we obviously have $dr = d\phi = 0$ (S is fixed on the surface and does not move), and we obtain

$$d\tau = \sqrt{1 - \frac{r_s}{r}}dt, \tag{13}$$

where the radius has to be evaluated at $r = R$. In case of F, we still have $d\phi = 0$, but now we must consider the astronaut's motion along the radial geodesic. Setting the angular momentum h to zero, the radial equation of motion in the Schwarzschild metric reads

$$\dot{r}^2 = c^2\left(k^2 - 1 + \frac{r_s}{r}\right), \tag{14}$$

with the dot representing the derivative w.r.t. the proper time τ. As F has been freely falling from infinity, his velocity must vanish for $r \to \infty$, which gives $k = 1$. Using

$$\alpha\dot{t} = k, \qquad \alpha = 1 - \frac{r_s}{r}, \tag{15}$$

we find that

$$dr^2 = \frac{r_s}{r}\alpha^2 c^2 dt^2, \tag{16}$$

and inserting the above back into equation (12), we finally obtain

$$d\tau = \alpha dt = \left(1 - \frac{r_s}{r}\right)dt, \tag{17}$$

where the radius can be assumed as constant ($r = R$) if the motion during the call is negligible (this is just our assumption). As expected, the additional motion of F leads to an increased time delay.

Finally let us consider the astronaut C who is moving in a circular orbit. Now we have $dr = 0$ and $d\phi \neq 0$, with

$$r^2\dot{\phi} = h. \tag{18}$$

Evidently, the circular orbit requires $\dot{r} = 0$ as well as that our astronaut is sitting in a minimum of the effective Schwarzschild potential, which, by definition, must occur at $r = R$. Thus, using the above conditions, the radial equation of motion and its derivative (minimum corresponds to extremal value, $dV_{eff}/dr = 0$), respectively, we obtain the following two relations:

$$k^2 = \left(1 + \frac{h^2}{c^2R^2}\right)\alpha, \qquad h^2 = c^2 r_s R\left(2 - \frac{3r_s}{R}\right)^{-1} \tag{19}$$

Substituting k and h in equations (15) and (18), we arrive at an expression for $R^2 d\phi^2$:

$$R^2 d\phi^2 = \frac{r_s}{R}\left(2 - \frac{3r_s}{R}\right)^{-1}\left(1 + \frac{r_s R\left(2 - \frac{3r_s}{R}\right)^{-1}}{R^2}\right)^{-1}\left(1 - \frac{r_s}{R}\right)c^2 dt^2 \tag{20}$$

In principal, the above could be inserted into equation (12) to give the final result. However, the current form is not suitable for comparison to the other cases. Thus, let us expand (20) to first order in r_s/R:

$$R^2 d\phi^2 = \frac{1}{2}\frac{r_s}{R}c^2 dt^2. \tag{21}$$

This yields

$$d\tau = \left(1 - \frac{3}{2}\frac{r_s}{R}\right)dt. \tag{22}$$

As we can see, the circular motion of C leads to a slightly bigger time delay. Interestingly, this is always true. Even the fact that the real relation between $d\tau$ and dt (based on equation (20)) creates a smaller effect than in case of F for $R \leq 1.5r_s$ does not change anything as the smallest allowed circular orbit must satisfy $R \geq 3r_s$ (Why?)

In the general case, one would have to consider that the travel time of light rays sent out by each astronaut may differ depending on the actual emission position **(different distances and angles w.r.t. their own motion to the receiver), i.e. moving astronauts will send information from different positions during their call. In particular, the Doppler effect causes the signal of astronauts moving towards the receiver to appear more frequent, while it becomes less frequent for astronauts moving away.** Additionally, one must consider the gravitational time delay (the propagation of light through curved space is delayed, similar to the case of light propagation through an optically thick medium). Also, note that the time dilatation between the reference frames will become more difficult to compute, e.g. in case of the radially moving observer $r = r(t)$ (see equation (17)).

Cleary, the observed signal on Earth will be redshifted. Neglecting any contribution due to cosmological expansion, there will be two effects: gravitational redshift and the Doppler shift, the latter being caused by a possible relative motion of sender and receiver. Therefore, if one does not want to risk losing contact to the astronauts, the receiver's calibration should take these redshift effects into account.