General Relativity

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Mathematical references: RHB = Riley, Hobson and Bence, Mathematical Methods for Physics and Engineering.

1 The Equivalence Principle

1.1 Mass in Newtonian Physics

One usually associates a unique *mass* with an object, but in fact mass appears 3 times in Newtonian physics, as *active gravitational*, *passive gravitational* and *inertial* mass:

- $\mathbf{g} = -Gm_{ga}\hat{\mathbf{r}}/r^2$
- $\mathbf{F} = m_{gp}\mathbf{g}$
- $\mathbf{F} = d(m_I\mathbf{v})/dt$

Newton's 3rd law ensures that $m_{ga} = m_{gp}$, since $\mathbf{F}_1 = -\mathbf{F}_2 \Rightarrow Gm_{ga1}m_{gp2}/r^2 = Gm_{ga2}m_{gp1}/r^2$, and hence

$$\frac{m_{ga1}}{m_{gp1}} = \frac{m_{ga2}}{m_{gp2}} \tag{1}$$

and the value of G can be adjusted so that the active and gravitational masses are not just proportional, but equal.

1.2 Eötvös experiments ($\sim$ 1890)

Torsion balance, with masses of different materials, but same gravitational mass m_g (checked with spring balance!). Force on each is

$$\mathbf{F} = m_g\mathbf{g} - m_I\boldsymbol{\Omega} \wedge (\boldsymbol{\Omega} \wedge \mathbf{r}) \tag{2}$$

where the last term is the inertial mass times the centripetal acceleration (or centrifugal force if it is placed on this side). Small torques would be detectable by swinging of wire. No torque was observed, which implies that, since the m_g are equal, so too are the inertial masses. Hence all three masses are the same, but there is no compelling reason in Newtonian physics why this should be so.

Also,

GRAVITY IS A 'KINEMATIC' (OR 'INERTIAL') FORCE, STRICTLY PROPORTIONAL TO THE MASS ON WHICH IT ACTS.

(so is centrifugal force).

1.2.1 Kinematic/Inertial forces can be transformed away

Consider a bee in a freely-falling lift without windows. Its equation of motion is

$$m\frac{d^2\mathbf{x}}{dt^2} = m\mathbf{g} + \mathbf{F} \tag{3}$$

where $\mathbf{F}$ here represents non-gravitational forces. To transform away gravity, move to the rest frame of the lift. i.e. make the non-Galilean spacetime transformation

$$\begin{aligned} \mathbf{x}' &= \mathbf{x} - \frac{1}{2}\mathbf{g}t^2 \\ t' &= t \end{aligned} \tag{4}$$

Then

$$m\frac{d^2\mathbf{x}'}{dt^2} = m\frac{d^2\mathbf{x}}{dt^2} - m\mathbf{g} = \mathbf{F} \tag{5}$$

In other words, the bee experimenter, who measures coordinates with respect to the lift, finds that Newton's laws are obeyed, *but does not detect the gravitational field.*

This argument is fine if $\mathbf{g}$ is uniform and time-independent. If it is not uniform, then a *large* lift can detect it, through the tidal forces which would, for example, draw together two bees in the Earth's gravitational field:

Thus we can remove gravity in a sufficiently small region (formally, if one specifies the accuracy with which one requires gravity to be removed, then one can define a 'sufficiently small region' within which this can be achieved; it will depend, of course, on the gradients in the gravitational field).

End of Lecture 1

This leads us to the *WEAK EQUIVALENCE PRINCIPLE*:

AT ANY POINT IN SPACETIME IN AN ARBITRARY GRAVITATIONAL FIELD, IT IS POSSIBLE TO CHOOSE A 'LOCALLY-INERTIAL FRAME' IN WHICH THE LAWS OF MOTION ARE THE SAME AS IF GRAVITY WERE ABSENT.

Einstein extended this to all of the laws of physics, to form the *STRONG EQUIVALENCE PRINCIPLE*:

IN A LOCALLY-INERTIAL FRAME, ALL OF SPECIAL RELATIVITY APPLIES.

(Hereafter referred to as 'EP').

Note that there are *infinitely many* LIFs at any spacetime point, all related by Lorentz transformations.

2 Gravitational Forces

Consider a freely-moving particle in a gravitational field. According to the EP, there is a coordinate system $\xi^\alpha = (ct, \mathbf{x})$ in which the particle follows a straight world line. It is convenient to write the world line *parametrically*, as a function, for example, of the proper time, $\xi^\alpha(\tau)$. A straight world line has

$$\boxed{\frac{d^2\xi^\alpha}{d\tau^2} = 0} \tag{6}$$

(Small exercise: show that this implies the 3 equations $\xi^i(t) = A_i t + B_i$ for constants A_i and B_i. This sort of unpacking is quite common.) Note that $c^2 d\tau^2 = \eta_{\alpha\beta} dx^\alpha dx^\beta = c^2 dt^2 - d\mathbf{x}^2$ is the square of the proper time interval, and

$$\eta_{\alpha\beta} = \begin{pmatrix} 1 & . & . & . \\ . & -1 & . & . \\ . & . & -1 & . \\ . & . & . & -1 \end{pmatrix} \tag{7}$$

is the metric of Minkowski spacetime. Note that Greek indices will run from 0 to 3, and Latin indices will run from 1 to 3. The Einstein summation convention applies unless otherwise stated.

Now consider *any* other coordinate system, which may be rotating, accelerating or whatever, in which the particle coordinates are $x^\mu(\tau)$. (The frame may, for example, be the 'lab rest frame'). Using the **chain rule** (RHB section 4.5)

$$d\xi^\alpha = \frac{\partial \xi^\alpha}{\partial x^\mu} dx^\mu \tag{8}$$

the SR equation of motion (6) becomes (note $d/d\tau = (dx^\mu/d\tau)(\partial/\partial x^\mu)$)

$$\begin{aligned} \frac{d}{d\tau}\left(\frac{\partial \xi^\alpha}{\partial x^\mu}\frac{dx^\mu}{d\tau}\right) &= 0 \\ \Rightarrow \frac{\partial \xi^\alpha}{\partial x^\mu}\frac{d^2x^\mu}{d\tau^2} + \frac{dx^\mu}{d\tau}\frac{d}{d\tau}\left(\frac{\partial \xi^\alpha}{\partial x^\mu}\right) &= 0 \\ \Rightarrow \frac{\partial \xi^\alpha}{\partial x^\mu}\frac{d^2x^\mu}{d\tau^2} + \frac{\partial^2 \xi^\alpha}{\partial x^\nu \partial x^\mu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} &= 0 \end{aligned} \tag{9}$$

We see that the acceleration will be zero if the ξ^α are *linear* functions of the new coordinates x^μ, since $\partial^2\xi^\alpha/\partial x^\nu \partial x^\mu$ then vanishes. This is the case for Lorentz transformations. For a general transformation this term is non-zero. (Note some subtleties in the above discussion: there is a general transformation between coordinate systems ξ^α and x^μ, so the partial derivatives exist everywhere. The specific path of the particle is $\xi^\alpha(\tau)$ or $x^\alpha(\tau)$, thus the appearance of total derivatives of these quantities w.r.t. τ).

To find the acceleration in the new frame, multiply by $\partial x^\lambda/\partial \xi^\alpha$ and use the **product rule** (which follows directly from the chain rule (8))

$$\frac{\partial x^\lambda}{\partial \xi^\alpha}\frac{\partial \xi^\alpha}{\partial x^\mu} = \delta^\lambda_\mu \tag{10}$$

where the right hand side is the Kronecker delta (=1 if $\lambda = \mu$ and zero otherwise).

This gives us the equation for a free particle, or the *GEODESIC EQUATION*:

$$\boxed{\frac{d^2x^\lambda}{d\tau^2} + \Gamma^\lambda{}_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0} \tag{11}$$

where $\Gamma^\lambda{}_{\mu\nu}$ is the *AFFINE CONNECTION*:

$$\Gamma^\lambda{}_{\mu\nu} \equiv \frac{\partial x^\lambda}{\partial \xi^\alpha}\frac{\partial^2 \xi^\alpha}{\partial x^\nu \partial x^\mu} \tag{12}$$

Note that Γ is symmetric in its lower indices, and, for future reference, it is *not* a tensor.

Note this is useful *conceptually*, as it comes directly from the EP. For practical purposes, this is rarely how one would actually *compute* Γ.

Further note that the proper time interval can be written in terms of dx^μ:

$$\begin{aligned} c^2 d\tau^2 &= \eta_{\alpha\beta} d\xi^\alpha d\xi^\beta = \eta_{\alpha\beta}\left(\frac{\partial \xi^\alpha}{\partial x^\mu} dx^\mu\right)\left(\frac{\partial \xi^\beta}{\partial x^\nu} dx^\nu\right) \\ \Rightarrow c^2 d\tau^2 &= g_{\mu\nu} dx^\mu dx^\nu \end{aligned} \tag{13}$$

where $g_{\mu\nu}$ is the *METRIC TENSOR* (note it is symmetric):

$$\boxed{g_{\mu\nu} \equiv \frac{\partial \xi^\alpha}{\partial x^\mu}\frac{\partial \xi^\beta}{\partial x^\nu}\eta_{\alpha\beta}} \tag{14}$$

(Example: Schwarzschild metric with spherical coordinates $x^\mu = (ct, r, \theta, \phi)$ and interval $c^2 d\tau^2 = c^2 dt^2(1 - 2GM/rc^2) - dr^2/(1 - 2GM/rc^2) - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2$ has non-zero components $g_{00} = -g_{11}^{-1} = 1 - 2GM/rc^2; g_{22} = -r^2; g_{33} = -r^2 \sin^2\theta$.)

2.1 Massless Particles

For massless particles, we cannot use $d\tau$, since it is zero. Instead we can use $\sigma \equiv \xi^0$ (ct in the LIF). Following similar logic, the condition $d^2\xi^\alpha/d\sigma^2 = 0$ becomes

$$\frac{d^2 x^\lambda}{d\sigma^2} + \Gamma^\lambda{}_{\mu\nu}\frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\sigma} = 0 \tag{15}$$

In neither this nor the massive particle case do we need to know what σ or τ are explicitly, since we have 4 equations to solve for e.g. $x^\mu(\tau)$, and τ can be eliminated to obtain the 3 equations $\mathbf{x}(t)$.

We see that whenever the term involving the affine connection is non-zero, the particle is accelerated. Since the acceleration is independent of the particle mass, the associated force is kinematic, and is interpreted as gravity (although in special cases it may be given a different name, such as centrifugal force). Note that we can *create* gravity by simply choosing a coordinate system in which Γ does not vanish.

End of Lecture 2

3 Motion of particles in a metric: $g_{\mu\nu}$ as gravitational potentials

The metric $g_{\mu\nu}$ is obtained from the solution of Einstein's field equations, which we will come to later. It controls the motion of particles, through the affine connection, as follows. i.e. Given a metric $g_{\mu\nu}$, how do particles move?

From (14),

$$g_{\mu\nu} \equiv \frac{\partial \xi^\alpha}{\partial x^\mu}\frac{\partial \xi^\beta}{\partial x^\nu}\eta_{\alpha\beta} \tag{16}$$

we have

$$\frac{\partial g_{\mu\nu}}{\partial x^\lambda} = \frac{\partial^2 \xi^\alpha}{\partial x^\lambda \partial x^\mu}\frac{\partial \xi^\beta}{\partial x^\nu}\eta_{\alpha\beta} + \frac{\partial \xi^\alpha}{\partial x^\mu}\frac{\partial^2 \xi^\beta}{\partial x^\lambda \partial x^\nu}\eta_{\alpha\beta} \tag{17}$$

and from the definition of the affine connection (12), we see that

$$\frac{\partial^2 \xi^\alpha}{\partial x^\nu \partial x^\mu} = \Gamma^\lambda{}_{\mu\nu}\frac{\partial \xi^\alpha}{\partial x^\lambda} \tag{18}$$

so (17) becomes

$$\frac{\partial g_{\mu\nu}}{\partial x^\lambda} = \Gamma^\rho{}_{\lambda\mu} \frac{\partial \xi^\alpha}{\partial x^\rho} \frac{\partial \xi^\beta}{\partial x^\nu} \eta_{\alpha\beta} + \Gamma^\rho{}_{\lambda\nu} \frac{\partial \xi^\alpha}{\partial x^\mu} \frac{\partial \xi^\beta}{\partial x^\rho} \eta_{\alpha\beta} \tag{19}$$

Using (14), this simplifies to

$$\frac{\partial g_{\mu\nu}}{\partial x^\lambda} = \Gamma^\rho{}_{\lambda\mu} g_{\rho\nu} + \Gamma^\rho{}_{\lambda\nu} g_{\mu\rho} \tag{20}$$

Now relabel indices: first $\mu \leftrightarrow \lambda$:

$$\frac{\partial g_{\lambda\nu}}{\partial x^\mu} = \Gamma^\rho{}_{\mu\lambda} g_{\rho\nu} + \Gamma^\rho{}_{\mu\nu} g_{\lambda\rho} \tag{21}$$

Second: $\nu \leftrightarrow \lambda$:

$$\frac{\partial g_{\mu\lambda}}{\partial x^\nu} = \Gamma^\rho{}_{\nu\mu} g_{\rho\lambda} + \Gamma^\rho{}_{\nu\lambda} g_{\mu\rho} \tag{22}$$

Add the first two of these, and subtract the last, and use the symmetry of Γ w.r.t. its lower indices, to get

$$\frac{\partial g_{\mu\nu}}{\partial x^\lambda} + \frac{\partial g_{\lambda\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\lambda}}{\partial x^\nu} = 2\Gamma^\rho{}_{\lambda\mu} g_{\rho\nu} \tag{23}$$

Now we *define* the **inverse of the metric tensor** as $g^{\sigma\rho}$, by

$$\boxed{g^{\sigma\rho} g_{\rho\nu} \equiv \delta^\sigma{}_\nu} \tag{24}$$

($g^{\sigma\rho}$ and $g_{\sigma\rho}$ are both symmetric). Hence

$$\boxed{\Gamma^\sigma{}_{\lambda\mu} \quad = \quad \frac{1}{2} g^{\nu\sigma} \left\{ \frac{\partial g_{\mu\nu}}{\partial x^\lambda} + \frac{\partial g_{\lambda\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\lambda}}{\partial x^\nu} \right\}.} \tag{25}$$

Note that the affine connections are sometimes written as $\left\{ {\sigma \atop \lambda\mu} \right\}$ and called *Christoffel Symbols.*

We see that the gravitational term in the geodesic equation depends on the *gradients* of $g_{\mu\nu}$, justifying their description as gravitational *potentials* (cf $\mathbf{g} = -\nabla\phi$ in Newtonian gravity). Note that there are 10 potentials, instead of one in Newtonian physics - no-one said GR was going to be easy! (Why 10 and not 16?)

If we can solve Einstein's equations for $g_{\mu\nu}(x^\alpha)$, we can solve the geodesic equation for the orbit. In general, this is the way to proceed, but if the problem has some symmetry to it, then a variational approach is easier (see chapter 6).

4 Newtonian Limit

If speeds are $\ll c$, and the gravitational field is weak and stationary, then the geodesic equation (11)

$$\frac{d^2 x^\lambda}{d\tau^2} + \Gamma^\lambda{}_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = 0 \tag{26}$$

can be approximated by ignoring the $d\mathbf{x}/d\tau$ terms in comparison with $d(ct)/d\tau$. Then

$$\frac{d^2 x^\lambda}{d\tau^2} + \Gamma^\lambda{}_{00} c^2 \left(\frac{dt}{d\tau} \right)^2 \simeq 0 \tag{27}$$

For a stationary field, $\partial g_{\mu\nu}/\partial t = 0$, so the affine connection (12) is

$$\Gamma^\lambda{}_{00} = \frac{1}{2} g^{\nu\lambda} \left\{ \frac{\partial g_{0\nu}}{\partial x^0} + \frac{\partial g_{0\nu}}{\partial x^0} - \frac{\partial g_{00}}{\partial x^\nu} \right\} = -\frac{1}{2} g^{\nu\lambda} \frac{\partial g_{00}}{\partial x^\nu} \tag{28}$$

For a *weak field*, we write

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{29}$$

and assume $|h_{\mu\nu}| \ll 1$. To first order in h,

$$\Gamma^{\lambda}{}_{00} = -\frac{1}{2}\eta^{\nu\lambda}\frac{\partial h_{00}}{\partial x^{\nu}} \tag{30}$$

and only the spatial parts ($\nu = 1, 2, 3$) of η survive (why?), and these have the value -1 along the diagonal ($\nu = \lambda$) and zero otherwise. Numerically this is a negative delta function:

$$\eta^{\nu\lambda}\frac{\partial h_{00}}{\partial x^{\nu}} = -\delta^{\nu}{}_{\lambda}\frac{\partial h_{00}}{\partial x^{\nu}} = -\frac{\partial h_{00}}{\partial x^{\lambda}} \tag{31}$$

The geodesic equation (27) then becomes

$$\frac{d^2x^{\lambda}}{d\tau^2} = -\frac{1}{2}c^2\left(\frac{dt}{d\tau}\right)^2\frac{\partial h_{00}}{\partial x^{\lambda}} \tag{32}$$

Vectorially, for the spatial parts,

$$\frac{d^2\mathbf{x}}{d\tau^2} = -\frac{1}{2}c^2\left(\frac{dt}{d\tau}\right)^2\nabla h_{00} \tag{33}$$

For speeds $\ll c$ and weak fields, $dt/d\tau \simeq 1$, and comparing with the Newtonian result $d^2\mathbf{x}/dt^2 = -\nabla\varphi$, we conclude that

$$h_{00} = \frac{2\varphi}{c^2} \tag{34}$$

plus a constant, which we take to be zero if we follow the convention that $\varphi \to 0$ far from any masses (where the metric approaches that of SR and $h \to 0$). Hence, in the *weak-field limit*,

$$\boxed{g_{00} \quad = \quad 1 + \frac{2\varphi}{c^2}} \tag{35}$$

which is consistent with our earlier assertion that $g_{\mu\nu}$ can be regarded as potentials.

End of Lecture 3

5 Time

What does it mean that g_{00} is not unity? *Events* are *labelled* by four coordinates (e.g. ct, r, θ, ϕ). It doesn't necessarily mean that t is the time measured by a clock at a specific location. We know this already, since if the clock is moving, there will be Doppler-type effects. The *proper time* is the time elapsed on a clock comoving with the object in question, and depends on its location, and its motion. We can get the proper time τ by using

$$ds^2 = c^2 d\tau^2 = g_{\mu\nu}dx^{\mu}dx^{\nu}. \tag{36}$$

So for a *stationary* clock (whose spatial coordinates are fixed, so $dx^i = 0,\ i = 1, 2, 3$),

$$ds^2 = c^2 d\tau^2 = g_{00}c^2 dt^2 \tag{37}$$

so that $d\tau = \sqrt{g_{00}}dt$. In the weak-field case, where $g_{00} = 1 + \frac{2\varphi}{c^2}$,

$$d\tau \simeq \left(1 + \frac{\varphi}{c^2}\right)dt \tag{38}$$

and we see that t coincides with τ only if $\varphi = 0$. So t is the *time elapsed on a stationary clock at infinity*, if we adopt the common convention that $\varphi = 0$ at infinity. The time elapsed on a stationary clock in a potential φ is not t - it runs at a different rate. We also see that *clocks run slow in a potential well* ($\varphi < 0$).

5.1 Gravitational redshift

We have a stationary light emitter at position $\mathbf{x}_1$ and a stationary observer at $\mathbf{x}_2$. If the emitter emits photons of frequency ν_1, what is the observed frequency?

Consider 2 events: emission of beginning of cycle of photon, and emission of next cycle. Since this is a light signal, $ds^2 = 0$, which gives a quadratic equation for dt in terms of dx^i (whose coefficients are time-independent if the metric is stationary). Thus we can integrate to find how long it takes the photon to reach $\mathbf{x}_2$. The (coordinate) time taken by the two events will be the same, so it is clear that coordinate time intervals at emission and reception are the same. i.e.

$$\frac{d\tau_1}{\sqrt{g_{00}(\mathbf{x}_1)}} = dt = \frac{d\tau_2}{\sqrt{g_{00}(\mathbf{x}_2)}} \tag{39}$$

so

$$\frac{d\tau_1}{d\tau_2} = \sqrt{\frac{g_{00}(\mathbf{x}_1)}{g_{00}(\mathbf{x}_2)}} \tag{40}$$

so the relation between the emitted and the observed frequencies is

$$\frac{\nu_2}{\nu_1} = \frac{d\tau_1}{d\tau_2} = \sqrt{\frac{g_{00}(\mathbf{x}_1)}{g_{00}(\mathbf{x}_2)}} \tag{41}$$

In weak fields, $g_{00} \simeq 1 + 2\varphi/c^2$, (equation 35) so to $O(\varphi)$,

$$\frac{\nu_2}{\nu_1} \simeq \sqrt{\frac{\left(1 - \frac{2\varphi_2}{c^2}\right)}{\left(1 - \frac{2\varphi_1}{c^2}\right)}} \simeq 1 - \frac{\varphi_2}{c^2} + \frac{\varphi_1}{c^2} \tag{42}$$

and the *gravitational redshift* is

$$z_{grav} \equiv \frac{\Delta\nu}{\nu} = \frac{\nu_1 - \nu_2}{\nu_1} = \frac{\varphi_2 - \varphi_1}{c^2}. \tag{43}$$

This is small for most astronomical bodies ($\sim 10^{-6}$ for the Sun), and often masked by Doppler effects e.g. convection in Sun, which gives systematic effects which are larger than this.

6 Variational formulation of GR

Dynamical equations in the form of differential equations may be written as a variational principle. We will see later that this approach offers some advantages in calculation.

6.1 Variational Principle

A particle follows a worldline between spacetime points A and B. Letting p be a parameter which increases monotonically along the path, the proper time elapsed is

$$c\tau_{AB} = c\int_A^B d\tau = c\int_A^B \frac{d\tau}{dp}dp = \int_A^B L(x^\mu, \dot{x}^\mu)dp \tag{44}$$

where (using (13))

$$L = c\frac{d\tau}{dp} = \sqrt{g_{\mu\nu}\frac{dx^\mu}{dp}\frac{dx^\nu}{dp}} \tag{45}$$

and $\dot{x}^\mu = dx^\mu/dp$, and p is a parameter which increases monotonically along the world line.

If the end-points A and B are fixed, then L satisfies the *Euler-Lagrange equations*

$$\frac{\partial L}{\partial x^\mu} - \frac{d}{dp}\left(\frac{\partial L}{\partial \dot{x}^\mu}\right) = 0 \qquad (46)$$

(RHB section 20.1).

In tutorial 1: From the Euler-Lagrange equations, prove that the statement that τ_{AB} is stationary is equivalent to the geodesic equation (11). Take $p = \tau$.

The square root in the EL equation (46) is a nuisance computationally, so consider instead L^2. Using the EL equation,

$$\frac{\partial L^2}{\partial x^\mu} - \frac{d}{dp}\left(\frac{\partial L^2}{\partial \dot{x}^\mu}\right) = -2\frac{dL}{dp}\frac{\partial L}{\partial \dot{x}^\mu} \qquad (47)$$

Now we can make the r.h.s. zero by noting that since $L = cd\tau/dp$, we have $dL/dp = cd^2\tau/dp^2 = 0$ *if* we choose p to be what is called an *affine parameter*, which is any parameter which increases *linearly with* τ.

So for affine parameters p,

$$\boxed{\frac{\partial L^2}{\partial x^\mu} - \frac{d}{dp}\left(\frac{\partial L^2}{\partial \dot{x}^\mu}\right) \quad = \quad 0 \qquad \text{ELII}} \qquad (48)$$

Call this equation *ELII*. (As before, argument breaks down for photons, but can find parameters for which ELII still holds. We will use ELII extensively for computations).

7 Calculations I: the Affine Connections or Christoffel symbols

Can use (12) directly, but can be time-consuming since many may be zero if problem has a lot of symmetry. Quick computation is possible using ELII. Rule is:

- Write ELII for each variable
- Rearrange ELII to get $\ddot{x}^\sigma +$ something $\dot{x}^\mu \dot{x}^\nu = 0$
- Read off the affine connection $\Gamma^\sigma{}_{\mu\nu}$. (There may be more than one term)
- If $\mu \neq \nu$, divide by 2. (Why?[1])
- Any Γ not appearing are zero

Example: 2D surface of sphere of radius R (this example is not a spacetime example, just space, for illustration). Separation[2] between points with coordinates θ, ϕ separated by $d\theta, d\phi$ is

$$\begin{aligned} d\ell^2 &= R^2 d\theta^2 + R^2 \sin^2\theta \, d\phi^2 \\ \Rightarrow L^2 &= R^2\dot{\theta}^2 + R^2 \sin^2\theta \, \dot{\phi}^2 \end{aligned} \qquad (49)$$

Apply the ELII equations, first to θ:

$$\begin{aligned} \frac{\partial}{\partial \theta}\left(R^2\dot{\theta}^2 + R^2\sin^2\theta\,\dot{\phi}^2\right) - \frac{d}{dp}\left[\frac{\partial}{\partial\dot{\theta}}\left(R^2\dot{\theta}^2 + R^2\sin^2\theta\,\dot{\phi}^2\right)\right] &= 0 \\ \Rightarrow 2R^2\sin\theta\cos\theta\dot{\phi}^2 - 2R^2\ddot{\theta} &= 0 \\ \Rightarrow \ddot{\theta} - \sin\theta\cos\theta\,\dot{\phi}^2 &= 0 \end{aligned} \qquad (50)$$

[1]Because μ and ν are both dummy indices and summed over, so a term in the ELII equation with different indices comes from two terms in the sum.

Hence

$$\Gamma^{\theta}{}_{\phi\phi} = -\sin\theta\cos\theta \tag{51}$$

Second, to ϕ:

$$\begin{aligned}
\frac{\partial}{\partial\phi}\left(R^2\dot\theta^2 + R^2\sin^2\theta\,\dot\phi^2\right) - \frac{d}{dp}\left[\frac{\partial}{\partial\dot\phi}\left(R^2\dot\theta^2 + R^2\sin^2\theta\,\dot\phi^2\right)\right] &= 0 \\
\Rightarrow 2R^2\left(2\sin\theta\cos\theta\,\dot\theta\dot\phi + \sin^2\theta\,\ddot\phi\right) &= 0 \\
\Rightarrow \ddot\phi + 2\cot\theta\,\dot\theta\dot\phi &= 0 \qquad (52)
\end{aligned}$$

so $\Gamma^{\phi}{}_{\phi\theta} = \Gamma^{\phi}{}_{\theta\phi} = \cot\theta$.

It is convenient for later purposes to write as two 2×2 matrices $(\Gamma_\theta)^\alpha{}_\beta \equiv \Gamma^\alpha{}_{\theta\beta}$:

$$\Gamma_\theta = \begin{pmatrix} \cdot & \cdot \\ \cdot & \cot\theta \end{pmatrix} \qquad \Gamma_\phi = \begin{pmatrix} \cdot & -\sin\theta\cos\theta \\ \cot\theta & \cdot \end{pmatrix} \tag{53}$$

End of Lecture 4

8 Schwarzschild Metric

8.1 Preamble: Alternative labelling for 2-sphere

Previously we used spherical angles θ and ϕ to label the surface, giving (49). If we let $r \equiv R\theta$ (r is called the *geodesic distance*), then

$$d\ell^2 = dr^2 + R^2\sin^2\left(\frac{r}{R}\right)d\phi^2 \tag{54}$$

Here the first term is as simple as we can make it, but the second is complicated (except when $R \to \infty$, when we get the flat plane in polar coordinates $dr^2 + r^2 d\phi^2$).

There are infinitely-many alternatives. For example, we can try to make the coefficient of $d\phi^2$ as simple as possible, by taking $\rho \equiv R\sin(r/R)$, so $d\rho = \cos(r/R)dr = \sqrt{1-\rho^2/R^2}dr$, and

$$\boxed{d\ell^2 = \frac{d\rho^2}{1-\kappa\rho^2} + \rho^2\,d\phi^2} \tag{55}$$

where $\kappa \equiv 1/R^2$ is the curvature of the sphere. ρ is then called the *angular diameter distance*, defined so that the angle subtended by a rod of length $d\ell_\perp$ perpendicular to the line-of-sight is $d\phi = d\ell_\perp/\rho$. Note that (55) applies to spheres ($\kappa > 0$), flat surfaces ($\kappa = 0$), but also 'hyperbolic' negatively-curved surfaces ($\kappa < 0$), which cannot be visualised in 3D. Horse saddles have negative curvature, but curvature is not uniform.

8.2 Special Relativity metric in spherical coordinates

Metric is $ds^2 = c^2dt^2 - d\ell^2$, where the spatial part of the metric $d\ell^2 = dx^2 + dy^2 + dz^2$ in Cartesian coordinates, or $d\ell^2 = dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2$ in spherical coordinates (r, θ, ϕ). It is instructive to note that

$$d\ell^2 = dr^2 + r^2 d\psi^2 \tag{56}$$

where $d\psi^2 = d\theta^2 + \sin^2\theta d\phi^2$ is the square of the angle between radial lines separated by $(d\theta, d\phi)$.

The SR metric may therefore be written

$$ds^2 = c^2dt^2 - \left[dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right] \tag{57}$$

The Schwarzschild metric applies to a spherically-symmetric mass distribution, e.g. outside a (point) mass. Space around a mass is not Euclidean, so we cannot use the SR metric. We can exploit the symmetry to find the form of the metric (a detailed derivation is postponed until section 14). Firstly, the symmetry suggests the use of spherical polars. For r we will use the angular diameter distance, so the perpendicular part of the metric is $r^2 d\psi^2$ (this *defines* r). Because space is curved, we don't expect to see dr^2 or c^2dt^2 alone (remember we expect g_{00} to be modified by the spatially-dependent gravitational potential in weak fields), but rather

$$ds^2 = e(r)c^2dt^2 - \left[f(r)dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right] \tag{58}$$

By isotropy, e and f cannot depend on direction. If we assume that the metric is *stationary*, then they won't depend on t either (can drop this later). Far away, metric is SR, so $e, f \to 1$ as $r \to \infty$. Dimensional analysis $\Rightarrow e, f$ depend on $GM/(rc^2)$.

Einstein's field equations give

$$\boxed{ds^2 = \left(1 - \frac{2GM}{rc^2}\right)c^2dt^2 - \frac{dr^2}{1 - \frac{2GM}{rc^2}} - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)} \tag{59}$$

Notes:

- metric is exact
- coefficient of dt^2 agrees with our weak-field calculation when $r \gg GM/(c^2)$
- t is *coordinate time*, and runs at a rate of *stationary clocks at* ∞. The *proper time* elapsed, for a stationary clock at (r, θ, ϕ) is $d\tau = dt\sqrt{1 - 2GM/rc^2}$ (from EP: $ds^2 = c^2d\tau^2$ and spatial coordinates are unchanging).

9 Orbits in the Schwarzschild metric

Apply Euler-Lagrange equations (48) to Schwarzschild metric:

$$\frac{\partial L^2}{\partial x^\mu} - \frac{d}{dp}\left(\frac{\partial L^2}{\partial \dot{x}^\mu}\right) = 0 \tag{60}$$

As before, for matter particles, take $p = \tau$, for which $L^2 = c^2(d\tau/dp)^2 = c^2$ along the correct path. In general, L^2 is

$$L^2 = \left(1 - \frac{2GM}{rc^2}\right)c^2\dot{t}^2 - \frac{\dot{r}^2}{1 - \frac{2GM}{rc^2}} - r^2\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right) \tag{61}$$

where the dot indicates now $d/d\tau$. Writing $\alpha \equiv 1 - 2GM/(rc^2)$, then the ELII equations with variable $= t$, θ, and ϕ are:

$$\begin{aligned} -\frac{d}{dp}(2c^2\alpha\dot{t}) &= 0 \\ 2r^2\sin\theta\cos\theta\dot{\phi}^2 - \frac{d}{dp}(-2r^2\dot{\theta}) &= 0 \\ -\frac{d}{dp}(-2r^2\sin^2\theta\dot{\phi}) &= 0 \end{aligned} \tag{62}$$

(Don't do r this way - see later). The first of these gives

$$\boxed{\alpha\dot{t} = \text{constant} = k} \tag{63}$$

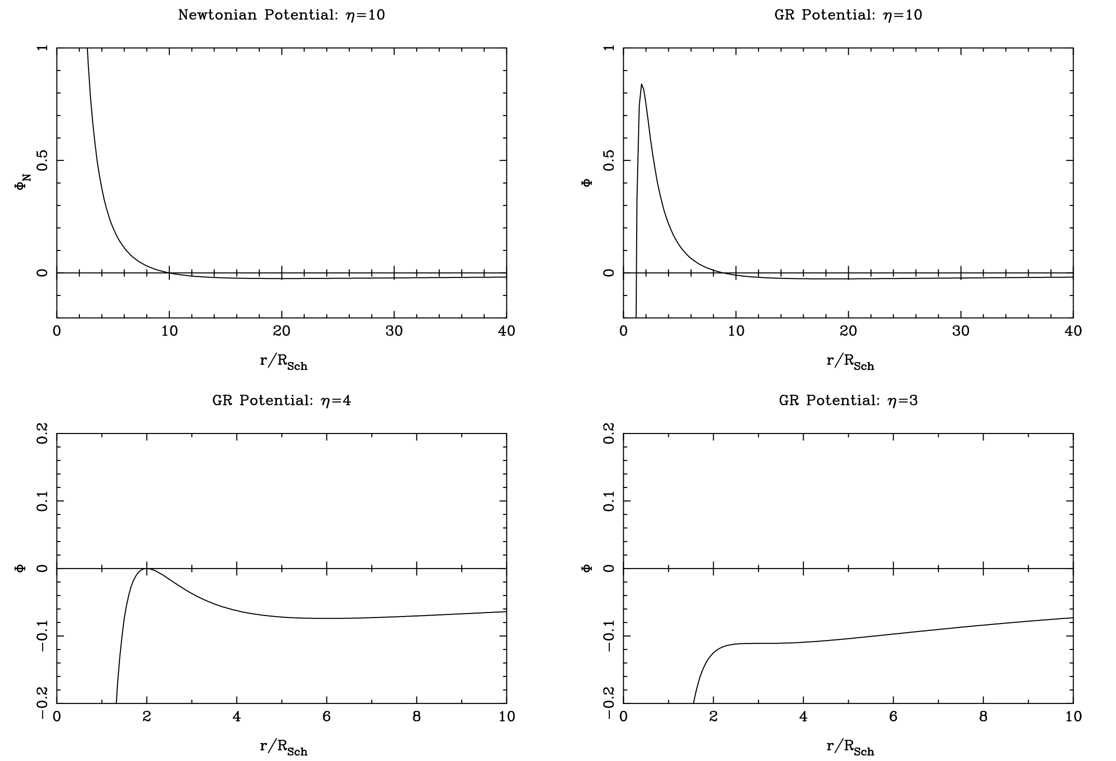

Figure 1: Effective potentials for Newton and GR gravity.

which expresses conservation of energy. Without loss of generality, define the orbit to lie in the equatorial plane, $\theta = \pi/2$ and $\dot{\theta} = 0$, so the third ELII equation gives

$$\boxed{r^2\dot{\phi} = \text{constant} = h} \tag{64}$$

which expresses conservation of angular momentum.

<u>End of Lecture</u> <u>5</u>

To get $\dot{r}$, we use the fact that $L^2 = c^2$ for a massive particle, so

$$\begin{aligned} c^2 &= c^2\alpha\frac{k^2}{\alpha^2} - \frac{\dot{r}^2}{\alpha} - r^2\frac{h^2}{r^4} \\ \rightarrow \dot{r}^2 + \alpha\frac{h^2}{r^2} &= c^2(k^2 - \alpha) = c^2k^2 - c^2 + \frac{2GM}{r} \end{aligned} \tag{65}$$

Compare this with Newtonian orbits,

$$\frac{d^2r}{dt^2} = -\frac{GM}{r^2} + \frac{v^2}{r} = -\frac{GM}{r^2} + \frac{h_N^2}{r^3} \tag{66}$$

where $h_N = vr$ is the Newtonian specific angular momentum. Multiplying by $2(dr/dt)$ and integrating gives

$$\boxed{\left(\frac{dr}{dt}\right)^2 + \frac{h_N^2}{r^2} - \frac{2GM}{r} = \text{constant}} \tag{67}$$

Compare the GR result (65) which can be cast as

$$\boxed{\dot{r}^2 + \frac{h^2}{r^2} - \frac{2GM}{r} - \frac{2GMh^2}{r^3c^2} = c^2(k^2 - 1) = \text{constant}} \tag{68}$$

So we see there is an *extra term* in the GR equations, but note different formal definitions of t, τ, r and h.

<u>End of Lecture</u> <u>5</u>

9.1 Effective potentials

Letting $R \equiv r/R_{Sch}$ where the denominator is the Schwarzschild radius $2GM/c^2$, and

$$\eta \equiv \frac{h^2}{R_{Sch}^2 c^2}, \tag{69}$$

($h \rightarrow h_N$ if Newtonian) then the Newtonian orbit equation can be written (where dot indicates d/dt here)

$$\frac{\dot{r}^2}{c^2} = -\left[\frac{\eta}{R^2} - \frac{1}{R}\right] + K \tag{70}$$

where K is a constant. The bracketed term is the *effective potential*: cf $\frac{1}{2}mv^2 + \phi = \text{constant} = K$. For Newtonian gravity, define

$$\Phi_N \equiv \frac{\eta}{R^2} - \frac{1}{R} \tag{71}$$

K is a constant of integration, which may be positive or negative (in which case the particle never reaches infinity). Note that $K \geq \Phi_N$ for physical solutions ($\dot{r}^2 > 0$).

The figure (panel 1) shows that bound ($K < 0$) and unbound ($K \geq 0$) orbits are possible. Note that *only if* $\eta = 0$ *can a particle reach* $r = 0$.

In GR, the effective potential is

$$\Phi \equiv \frac{\eta}{R^2} - \frac{1}{R} - \frac{\eta}{R^3} \tag{72}$$

The last term adds an attracting potential at small r. Orbits depend *qualitatively* on the value of η.

For large η (panel 2) shows that a particle can fall in, *even if* $\eta > 0$.

For $\eta = 4$, unbound orbits disappear (panel 3).

For $\eta = 3$, bound orbits disappear (panel 4). **LAST STABLE ORBIT** occurs at $R = 3$ (i.e. $r = 6GM/c^2$).

Problem 1: Compute the critical values of η (3 and 4).

9.2 Advance of perihelion of Mercury

Convenient to work with $u \equiv 1/r$. Hence

$$\dot{r} = \frac{dr}{d\tau} = \frac{dr}{d\phi}\frac{d\phi}{d\tau} = -\frac{1}{u^2}\frac{du}{d\phi}hu^2 = -h\frac{du}{d\phi} \tag{73}$$

The geodesic equation (68) becomes

$$h^2\left(\frac{du}{d\phi}\right)^2 + h^2u^2 - 2GMu - \frac{2GM}{c^2}h^2u^3 = c^2(k^2 - 1) \tag{74}$$

Differentiating w.r.t. ϕ and dividing by $2du/d\phi$ gives

$$\boxed{\frac{d^2u}{d\phi^2} + u = \frac{GM}{h^2} + \frac{3GM}{c^2}u^2} \tag{75}$$

<u>**End of Lecture** 6</u>

The (last) GR correction term is very small for Mercury's orbit, $\sim 10^{-7}$ of GM/h^2, so treat it as a perturbation. First, make dimensionless: define radius in terms of circular Newtonian radius of same h. i.e. let

$$U \equiv u\frac{h^2}{GM} \tag{76}$$

So

$$\frac{d^2U}{d\phi^2} + U = 1 + \epsilon U^2 \tag{77}$$

where $\epsilon \equiv 3G^2M^2/(c^2h^2)$. Expand $U = U_0 + U_1$, where U_0 is solution of

$$\frac{d^2U_0}{d\phi^2} + U_0 = 1 \tag{78}$$

i.e. $U_0 = 1 + A\cos\phi + B\sin\phi = 1 + e\cos\phi$ where e is the orbit eccentricity and we have removed B by choice of origin of ϕ. U_1 satifies

$$\frac{d^2U_1}{d\phi^2} + U_1 = \epsilon U_0^2 = \epsilon\left(1 + 2e\cos\phi + e^2\cos^2\phi\right) = \epsilon\left(1 + 2e\cos\phi + \frac{e^2}{2} + \frac{e^2}{2}\cos 2\phi\right). \tag{79}$$

The complementary function gives nothing new ($\propto U_0$); PI is

$$U_1 = A + B\phi\sin\phi + C\cos 2\phi \tag{80}$$

(extra ϕ because $\sin\phi$ is in CF). Solution (exercise) is

$$U_1 = \epsilon\left[\left(1 + \frac{e^2}{2}\right) + e\phi\sin\phi - \frac{e^2}{6}\cos 2\phi\right]. \tag{81}$$

Ignoring everything except the growing term $\propto \phi$,

$$\begin{aligned} U &\simeq 1 + e\cos\phi + \epsilon e\phi\sin\phi \\ &\simeq 1 + e\cos\left[\phi\left(1-\epsilon\right)\right] \end{aligned} \tag{82}$$

to $O(\epsilon)$. Orbit is periodic, but with period (in ϕ) of

$$\frac{2\pi}{1-\epsilon}. \tag{83}$$

The perihelion moves round through an angle $2\pi\epsilon$ per orbit. This works out at 43 arcseconds per century, for Mercury's orbit of $T = 88$ days, $r = 5.8\times 10^{10}m$ and $e = 0.2$.

End of Lecture 6

9.3 The bending of light round the Sun

Apply ELII to light, but use affine parameter p rather than proper time τ. As before

$$\begin{aligned} \alpha\dot{t} &= k = \text{constant} \\ r^2\dot{\phi} &= h = \text{constant} \end{aligned} \tag{84}$$

where $\dot{} = d/dp$. For light $L^2 = 0 \Rightarrow$

$$0 = \alpha c^2\dot{t}^2 - \frac{\dot{r}^2}{\alpha} - r^2\dot{\phi}^2 \tag{85}$$

Rearranging, and writing in terms of $u = 1/r$ as before,

$$h^2\left(\frac{du}{d\phi}\right)^2 = c^2k^2 - \alpha h^2u^2 = c^2k^2 - h^2u^2 + \frac{2GM}{c^2}h^2u^3 \tag{86}$$

Differentiating as before,

$$\boxed{\frac{d^2u}{d\phi^2} + u = \frac{3GM}{c^2}u^2} \tag{87}$$

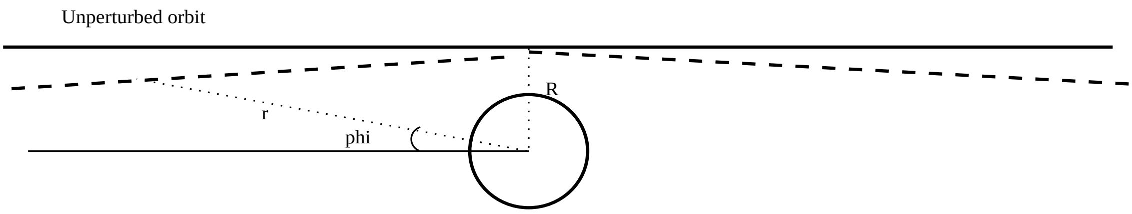

Figure 2: Light bending round the Sun.

Again we treat the r.h.s. as a perturbation to the orbit

$$u_0 = A \sin\phi + B \cos\phi = \frac{\sin\phi}{R} \tag{88}$$

where R is the distance of closest approach (equation is a straight line). Letting $u = u_0 + u_1$,

$$\frac{d^2 u_1}{d\phi^2} + u_1 = \frac{3GM}{c^2 R^2} \sin^2\phi = \frac{3GM}{2c^2 R^2}(1 - \cos 2\phi) \tag{89}$$

By inspection, the first-order solution is

$$u = \frac{\sin\phi}{R} + \frac{3GM}{2c^2 R^2}\left(1 + \frac{1}{3}\cos 2\phi\right) \tag{90}$$

At large distances, where ϕ is small, $\sin\phi \simeq \phi$, and $\cos 2\phi \simeq 1$, so

$$u \to \frac{\phi_{-\infty}}{R} + \frac{2GM}{c^2 R^2} \to 0 \tag{91}$$

i.e. $r \to -\infty$ at

$$\phi_{-\infty} = -\frac{2GM}{c^2 R} \tag{92}$$

Similarly, after the light has passed the source, it reaches infinite distance at

$$\phi_{+\infty} = \pi + \frac{2GM}{c^2 R} \tag{93}$$

so the total deflection is

$$\Delta\phi_{GR} = \frac{4GM}{c^2 R} \tag{94}$$

9.3.1 Newtonian argument

Treat photon as massive particle travelling at c, with $h = cR$. Then

$$\frac{d^2 u}{d\phi^2} + u = \frac{GM}{h^2} = \frac{GM}{c^2 R^2} \tag{95}$$

Appropriate solution is

$$u = \frac{\sin\phi}{R} + \frac{GM}{c^2 R^2} \tag{96}$$

and $u \to 0$ when $\phi \to -GM/(c^2 R)$ or $\pi + GM/(c^2 R)$. so the total Newtonian deflection is

$$\Delta\phi_{Newt} = \frac{2GM}{c^2 R} \tag{97}$$

exactly half the GR result.

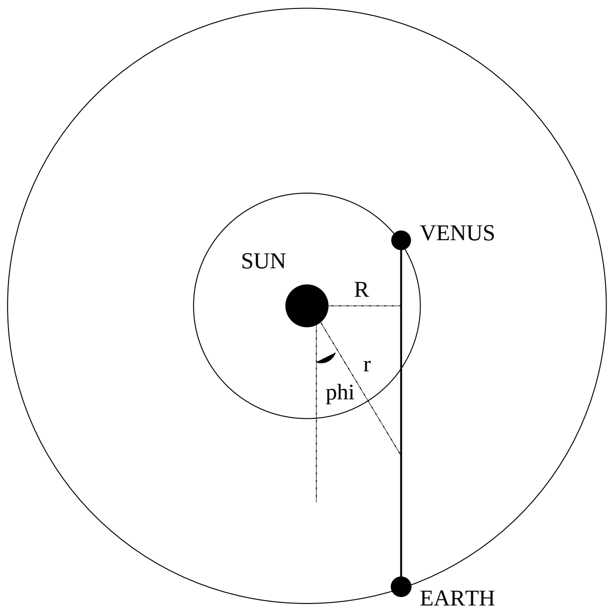

Figure 3: Radar time delay off Venus.

9.4 Time delay of light

Proposed by Shapiro (1964): bounce radar off Venus and measure time to return. We therefore need dt/dr. Use Schwarzschild metric in the equatorial plane and $ds^2 = 0$:

$$0 = \alpha c^2 dt^2 - \frac{dr^2}{\alpha} - r^2 d\phi^2 \tag{98}$$

Eliminate ϕ using the unperturbed (Newtonian) solution $r \sin\phi = R =$ distance of closest approach to the Sun. Hence

$$\begin{aligned} dr\sin\phi + r d\phi\cos\phi &= 0 \\ \Rightarrow r^2 d\phi^2 &= dr^2 \tan^2\phi \\ \Rightarrow r^2 d\phi^2 &= \frac{R^2 dr^2}{r^2 - R^2} \end{aligned} \tag{99}$$

Hence

$$c^2 dt^2 = \frac{dr^2}{\alpha^2} + \frac{r^2 d\phi^2}{\alpha} = dr^2 \left[\left(1 - \frac{2GM}{rc^2}\right)^{-2} + \left(1 - \frac{2GM}{rc^2}\right)^{-1} \frac{R^2}{r^2 - R^2} \right] \tag{100}$$

To first order in $GM/(rc^2)$ (after some algebra),

$$cdt = \pm \frac{rdr}{\sqrt{r^2 - R^2}} \left[1 + \frac{2GM}{rc^2} - \frac{GMR^2}{r^3 c^2} \right] \tag{101}$$

Integrating:

$$c\Delta t = \left[\sqrt{r^2 - R^2} + \frac{2GM}{Rc^2} \ln\left(\sqrt{r^2 - R^2} + r\right) - \frac{GM}{c^2}\left(r^2 - R^2\right)^2 \right]_{\mathrm{r(Earth)}}^{\mathrm{r(Venus)}} \tag{102}$$

Note that this is the *coordinate* time elapsed - the time elapsed on Earth is *not* Δt. To calculate it, we note that $ds^2 = c^2 d\tau^2$ where $d\tau$ is the (proper) time elapsed on Earth. It is related to Δt; how? We note that the Earth is not stationary, so there is a term in ds^2 which comes from the motion of the Earth (the $d\phi$ term). Thus

$$c^2 d\tau^2 = \left(1 - \frac{2GM}{r_{Earth} c^2}\right) c^2 dt^2 - r^2 d\phi^2$$

($dr = d\theta = 0$, near enough). Hence the proper time elapsed on Earth

$$d\tau = \sqrt{\alpha(r_{Earth}) - \frac{r^2}{c^2}\left(\frac{d\phi}{d\tau}\right)^2}\, dt.$$

The angular velocity of the Earth obeys (Newtonian approximation is good enough here) $(d\phi/d\tau)^2 = GM/r^3$, so the premultiplier of dt is

$$1 - \frac{2GM}{rc^2} - \frac{GM}{rc^2} = 1 - \frac{3GM}{rc^2}$$

and $\Delta\tau \simeq \left(1 - \frac{3GM}{2rc^2}\right)\Delta t$.

End of Lecture 7

10 The Route to Einstein's Field Equations

10.1 Principle of General Covariance

Alternative formulation of Equivalence Principle (same physical content).

Physical equation holds in a gravitational field if

- The equation holds in the absence of gravity.
- Equation is *Generally Covariant*. i.e. it *preserves its form* under general transformations $x \to x'$.

This is equivalent to the EP, since if the latter is obeyed, and if the equation is true in a local inertial frame, then it is true in a general system.

10.2 Correspondence Principle

GR should reduce to SR in the absence of gravity, and to Newtonian gravity in the weak-field, slow-speed limit.

11 Tensors

In order to construct equations which preserve their form (are covariant) under general coordinate transformations, we need to know how the quantities in the equations transform. There are certain objects, such as dx^μ which transform simply. These are called *tensors*.

0. Scalars

No index. Don't change at all. Example $d\tau$. Note, not all numbers are scalars: number density n is not, because of length contraction.

1. Vectors

(a) *Contravariant vectors* (index up) transform according to

$$V'^{\mu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} V^{\nu} \tag{103}$$

Example: dx^{μ}. Chain rule gives

$$dx'^{\mu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} dx^{\nu} \tag{104}$$

(b) *Covariant vectors* (index down) transform according to

$$U'_{\mu} = \frac{\partial x^{\nu}}{\partial x'^{\mu}} U_{\nu} \tag{105}$$

Example: if φ is a scalar field, then $\partial\varphi/\partial x^{\mu}$ is a covariant vector, since

$$\begin{aligned} d\varphi &= \frac{\partial \varphi}{\partial x^{\nu}} dx^{\nu} \\ \Rightarrow \frac{\partial \varphi}{\partial x'^{\mu}} &= \frac{\partial x^{\nu}}{\partial x'^{\mu}} \frac{\partial \varphi}{\partial x^{\nu}} \end{aligned} \tag{106}$$

QED.

2. Tensors (of arbitrary rank)

Scalars have rank 0, vectors have rank 1. A tensor with upper indices $\alpha, \beta \ldots$ and upper indices $\mu, \nu, \ldots$ transforms like a product of tensors of different types $U^{\alpha} V^{\beta} \ldots W_{\mu} X_{\nu} \ldots$. e.g.

$$T'^{\mu}{}_{\alpha\nu} = \frac{\partial x'^{\mu}}{\partial x^{\sigma}} \frac{\partial x^{\kappa}}{\partial x'^{\alpha}} \frac{\partial x^{\rho}}{\partial x'^{\nu}} T^{\sigma}{}_{\kappa\rho} \tag{107}$$

T can be contravariant (all indices up), covariant (all down), or mixed.

Example: the metric tensor

$$g_{\mu\nu} = \frac{\partial \xi^{\alpha}}{\partial x^{\mu}} \frac{\partial \xi^{\beta}}{\partial x^{\nu}} \eta_{\alpha\beta} \tag{108}$$

In a coordinate system x'^{μ},

$$\begin{aligned} g'_{\mu\nu} &= \frac{\partial \xi^{\alpha}}{\partial x'^{\mu}} \frac{\partial \xi^{\beta}}{\partial x'^{\nu}} \eta_{\alpha\beta} \\ &= \left(\frac{\partial \xi^{\alpha}}{\partial x^{\rho}} \frac{\partial x^{\rho}}{\partial x'^{\mu}} \right) \left(\frac{\partial \xi^{\beta}}{\partial x^{\sigma}} \frac{\partial x^{\sigma}}{\partial x'^{\nu}} \right) \eta_{\alpha\beta} \\ &= \frac{\partial x^{\rho}}{\partial x'^{\mu}} \frac{\partial x^{\sigma}}{\partial x'^{\nu}} g_{\rho\sigma} \end{aligned} \tag{109}$$

Can show (tutorial) that its inverse $g^{\mu\nu}$ is a contravariant tensor.

Note: not all things with indices are tensors. The affine connections $\Gamma^{\lambda}{}_{\mu\nu}$ are *not*.

An equation which is an equality of matched tensors will be generally covariant

since both sides transform as tensors. Note that indices must be in same places.

Tensor operations

1. Sum $T^{\mu}{}_{\nu} = aA^{\mu}{}_{\nu} + bB^{\mu}{}_{\nu}$ is a tensor if A and B are, and a and b are scalars.

2. Products. $T^{\alpha\beta}{}_{\gamma} = A^{\alpha} B^{\beta}{}_{\gamma}$ is a tensor if A and B are.

3. Contraction. If $T^{\alpha\beta}{}_{\mu\nu}$ is a tensor, then $T^{\alpha}{}_{\nu} \equiv T^{\alpha\beta}{}_{\beta\nu}$ is a tensor. Contraction must be over one upper and one lower index. Note the summation convention.

4. Raising and lowering indices. If $T^{\rho\mu}{}_{\sigma}$ is a tensor, then so is $T^{\rho}{}_{\nu\sigma} \equiv g_{\nu\mu} T^{\rho\mu}{}_{\sigma}$ by rules (2) and (3). So $g_{\mu\nu}$ *lowers* an index, and similarly $g^{\mu\nu}$ raises an index.

Note that in SR, the raising/lowering operation simply changes the sign of the spatial parts (if cartesian x, y, z coordinates are employed). In GR the operations are more complicated (as they are if, e.g. spherical polar coordinates are used), and defined by rule 4 above. Note that, in SR or GR, the versions of a vector with index up or down are both tensors, but have different transformation properties.

End of Lecture 8

11.1 Differentiating tensors

The derivative $\partial V^{\mu}/\partial x^{\lambda}$ of a tensor V^{μ} is not, in general, a tensor. Differentiating (103):

$$\begin{aligned} \frac{\partial V'^{\mu}}{\partial x'^{\lambda}} &= \frac{\partial}{\partial x'^{\lambda}} \left(\frac{\partial x'^{\mu}}{\partial x^{\nu}} V^{\nu} \right) = \frac{\partial x'^{\mu}}{\partial x^{\nu}} \frac{\partial V^{\nu}}{\partial x'^{\lambda}} + \frac{\partial x^{\rho}}{\partial x'^{\lambda}} \frac{\partial^2 x'^{\mu}}{\partial x^{\rho} \partial x^{\nu}} V^{\nu} \\ &= \frac{\partial x'^{\mu}}{\partial x^{\nu}} \frac{\partial x^{\rho}}{\partial x'^{\lambda}} \frac{\partial V^{\nu}}{\partial x^{\rho}} + \frac{\partial x^{\rho}}{\partial x'^{\lambda}} \frac{\partial^2 x'^{\mu}}{\partial x^{\rho} \partial x^{\nu}} V^{\nu} \end{aligned} \tag{110}$$

The first term is what we would expect if the derivative were a tensor. The last term destroys the tensor nature.

To create laws of nature (which are often differential equations) which are generally covariant, we need to define a derivative which is a tensor. This can be done: we define a 'covariant derivative':

$$\boxed{V^{\mu}{}_{;\lambda} \equiv \frac{\partial V^{\mu}}{\partial x^{\lambda}} + \Gamma^{\mu}{}_{\lambda\rho} V^{\rho}} \tag{111}$$

which *is* a tensor.

Proof:

We want an operator D_{α} such that

- $D_{\alpha} V^{\beta}$ is a tensor, if V^{β} is a vector
- $D_{\alpha} \to (\partial/\partial\xi^{\alpha})$ in a locally inertial frame ξ^{α}.

To be a tensor, we demand

$$\begin{aligned} D_{\alpha} V^{\beta} &= \frac{\partial x^{\beta}}{\partial x'^{\sigma}} \frac{\partial x'^{\rho}}{\partial x^{\alpha}} D'_{\rho} V'^{\sigma} \\ &= \frac{\partial x^{\beta}}{\partial x'^{\sigma}} \frac{\partial x'^{\rho}}{\partial x^{\alpha}} D'_{\rho} \left(\frac{\partial x'^{\sigma}}{\partial x^{\kappa}} V^{\kappa} \right) \end{aligned} \tag{112}$$

Now make x' the LIF ξ^{μ}, and $D'_{\rho} \to \partial/\partial\xi^{\rho}$.

$$D_{\alpha} V^{\beta} = \frac{\partial x^{\beta}}{\partial \xi^{\sigma}} \frac{\partial \xi^{\rho}}{\partial x^{\alpha}} \left(\frac{\partial}{\partial \xi^{\rho}} \frac{\partial \xi^{\sigma}}{\partial x^{\kappa}} V^{\kappa} + \frac{\partial \xi^{\sigma}}{\partial x^{\kappa}} \frac{\partial V^{\kappa}}{\partial \xi^{\rho}} \right) \tag{113}$$

$$D_{\alpha} V^{\beta} = \frac{\partial x^{\beta}}{\partial \xi^{\sigma}} \frac{\partial \xi^{\rho}}{\partial x^{\alpha}} \left(\frac{\partial^2 \xi^{\sigma}}{\partial x^{\kappa} \partial x^{\lambda}} \frac{\partial x^{\lambda}}{\partial \xi^{\rho}} V^{\kappa} + \frac{\partial \xi^{\sigma}}{\partial x^{\kappa}} \frac{\partial V^{\kappa}}{\partial \xi^{\rho}} \right) \tag{114}$$

Each term outside the bracket combines with one inside to give a delta function, leaving

$$\begin{aligned} D_\alpha V^\beta &= \frac{\partial \xi^\rho}{\partial x^\alpha}\frac{\partial V^\beta}{\partial \xi^\rho} + \frac{\partial x^\beta}{\partial \xi^\sigma}\frac{\partial^2 \xi^\sigma}{\partial x^\kappa \partial x^\alpha} V^\kappa \\ &= \frac{\partial V^\beta}{\partial x^\alpha} + \frac{\partial x^\beta}{\partial \xi^\sigma}\frac{\partial^2 \xi^\sigma}{\partial x^\kappa \partial x^\alpha} V^\kappa \\ \Rightarrow D_\alpha V^\beta &= V^\beta{}_{,\alpha} + \Gamma^\beta{}_{\kappa\alpha} V^\kappa \end{aligned} \tag{115}$$

Hence result.

Similarly, for covariant vectors,

$$\boxed{V_{\mu;\lambda} \equiv \frac{\partial V_\mu}{\partial x^\lambda} - \Gamma^\rho{}_{\mu\lambda} V_\rho} \tag{116}$$

is a tensor. For higher-rank tensors, each upper index adds a Γ term, and each lower index subtracts one. e.g.

$$T^\alpha{}_{\beta;\nu} = \frac{\partial T^\alpha{}_\beta}{\partial x^\nu} + \Gamma^\alpha{}_{\nu\rho} T^\rho{}_\beta - \Gamma^\rho{}_{\beta\nu} T^\alpha{}_\rho. \tag{117}$$

Covariant differentiation

- obeys Leibnitz rule ($(AB)_{;\alpha} = A_{;\alpha}B + A\,B_{;\alpha}$)
- commutes with contraction
- *doesn't* commute with itself: $V^\mu{}_{;\alpha;\beta} \neq V^\mu{}_{;\beta;\alpha}$.

Covariant derivative of $g_{\mu\nu}$ vanishes

Useful result:

$$g_{\mu\nu;\lambda} = \frac{\partial g_{\mu\nu}}{\partial x^\lambda} - \Gamma^\rho{}_{\lambda\mu} g_{\rho\nu} - \Gamma^\rho{}_{\lambda\nu} g_{\mu\rho} = 0 \tag{118}$$

where the last follows from (20). Similarly

$$g^{\mu\nu}{}_{;\lambda} = 0. \tag{119}$$

Thus covariant differentiation and raising/lowering commute. $(g^{\mu\nu}V_\nu)_{;\lambda} = g^{\mu\nu}V_{\nu;\lambda}$.

Algorithm for General Relativity

Covariant differentiation has 2 properties:

- It converts tensors to tensors
- It reduces to ordinary differentiation in the absence of gravity ($\Gamma = 0$).

So we will satisfy the Principle of General Covariance by the following rule:

Take the equations of Special Relativity, replace $\eta_{\alpha\beta}$ by $g_{\alpha\beta}$ and all derivatives by covariant derivatives

This is fine for everything except gravity, for which there is no SR theory. Newton's theory is 'action at a distance' and therefore unacceptable relativistically.

End of Lecture 9

11.2 Covariant differentiation along a curve

Vectors might be defined only along a worldline, rather than everywhere. e.g. momentum of particle $P^\mu(\tau)$ has no meaning except on the worldline $x^\mu(\tau)$. If $A^\mu(\tau)$ is such a tensor, then, by a similar argument,

$$\frac{DA^\mu}{D\tau} \equiv \frac{dA^\mu}{d\tau} + \Gamma^\mu{}_{\nu\lambda}\frac{dx^\lambda}{d\tau}A^\nu \tag{120}$$

is a tensor. Similarly, for covariant vectors,

$$\frac{DB_\mu}{D\tau} \equiv \frac{dB_\mu}{d\tau} - \Gamma^\lambda{}_{\mu\nu}\frac{dx^\nu}{d\tau}B_\lambda \tag{121}$$

is also a tensor.

11.3 Scalars

Note that the covariant derivative of a scalar field $\phi(x^\mu)$ is just $\partial\phi/\partial x^\nu$.

11.4 Parallel Transport

Is the total momentum of a system of particles conserved? Adding vectors in a curved spacetime is problematic, as we will see in this section. Indeed, even the question of whether two vectors are parallel is difficult. Two vectors can be compared *locally*, but how do you tell if two vectors are parallel if they are separated? They have to be moved to a common location, where their components can be compared. But how should one do this? It can be done, using the notion of *parallel transport*.

If we view a vector in a cartesian local inertial frame, we can move a vector A^μ from place-to-place by keeping the components unchanged. i.e. $dA^\mu/d\tau = 0$. This is not a tensor equation, so other observers may not agree that it is zero. However, in this frame the affine connections Γ vanish, so we also have

$$\frac{DA^\mu}{D\tau} = 0, \tag{122}$$

which *is* a covariant statement, so it is true in all frames. Hence in arbitrary frames,

$$\boxed{\frac{dA^\mu}{d\tau} = -\Gamma^\mu{}_{\nu\lambda}\frac{dx^\lambda}{d\tau}A^\nu} \tag{123}$$

This is the *equation of parallel transport*. Any vector may be parallel-transported along a curve $x^\mu(\tau)$ by requiring the covariant derivative to vanish. We can use this to determine if two vectors are parallel - parallel-transport one to the spacetime point of the other, and compare them. *However*, this needs the the path (worldline) is specified. To see this, consider parallel transport of a vector on a 2D spherical surface. A vector lying initially along the equator, and transported from the equator to the pole, back to the equator (at a point 90° away in longitude) and back to the starting point rotates through 90°. Therefore we conclude that the question 'Is the vector at the pole parallel to the initial vector at the starting point?' is ambiguous; in curved space it depends which path the vector is taken to compare.

11.5 Conserved quantities

Since we cannot unambiguously add vectors at different places in curved space, familiar concepts like the conservation of the total momentum of a system of particles are lost. For a single particle, there may be some conserved quantities, as we have discovered already for the Schwarzschild metric. *Globally*-conserved quantities are connected with symmetries of the metric. To show this in general, we look for the rate of change of the (covariant) components of the momentum along the worldline, $dp_\lambda/d\tau$ (the contravariant components don't have nice conservation laws). The momentum is $p^\mu \equiv m_0 U^\mu = m_0 \dot{x}^\mu$, where m_0 is the rest mass.

The geodesic equation (11) is

$$m_0 \dot{p}^\mu = -\Gamma^\mu{}_{\sigma\rho} p^\sigma p^\rho \neq 0 \tag{124}$$

Consider the covariant momentum, $p_\lambda = g_{\lambda\mu} p^\mu$:

$$\begin{aligned} \frac{dp_\lambda}{d\tau} &= \frac{dg_{\lambda\mu}}{d\tau} p^\mu + g_{\lambda\mu} \frac{dp^\mu}{d\tau} \\ &= \frac{\partial g_{\lambda\mu}}{\partial x^\alpha} \frac{dx^\alpha}{d\tau} p^\mu + \frac{g_{\lambda\mu}}{m_0} \left[-\frac{1}{2} g^{\mu\beta} \left(g_{\beta\sigma,\rho} + g_{\beta\rho,\sigma} - g_{\rho\sigma,\beta} \right) p^\sigma p^\rho \right] \end{aligned} \tag{125}$$

But $g_{\lambda\mu} g^{\mu\beta} = \delta^\beta_\lambda$, so

$$m_0 \frac{dp_\lambda}{d\tau} = g_{\lambda\mu,\alpha} p^\alpha p^\mu - \frac{1}{2} \left(g_{\lambda\sigma,\rho} + g_{\lambda\rho,\sigma} - g_{\rho\sigma,\lambda} \right) p^\sigma p^\rho \tag{126}$$

Relabelling the dummy indices ($\sigma \to \mu$; $\rho \to \alpha$ in the first term in brackets, then $\sigma \to \alpha$; $\rho \to \mu$ in the second) cancels all but the last term, leaving

$$m_0 \frac{dp_\lambda}{d\tau} = \frac{1}{2} g_{\rho\sigma,\lambda} p^\sigma p^\rho \tag{127}$$

which gives the important result that

If all of $g_{\mu\nu}$ are independent of some x^λ, then p_λ is constant along the worldline.

E.g. Schwarzschild metric is independent of t, which means p_t is constant. For axially-symmetric metrics, p_ϕ is constant. For Schwarzschild, this is $g_{\phi\alpha} p^\alpha = m_0 r^2 \sin\theta d\phi/d\tau$. We called this $(m_0)\, h$ before (in the equatorial plane).

In general, $g_{\mu\nu}$ has no symmetries, so there are no globally-conserved quantities.

11.6 Parallel transport and curvature

Consider 2D surfaces. The 2D surface of a cylinder has curvature when viewed in 3D, buts its *internal* properties are locally the same as a plane surface, as we can create it by bending a flat sheet without tearing or folding. The cylinder has *extrinsic* curvature, but no *intrinsic* curvature. A sphere, on the other hand, is *intrinsically* curved; one must tear or fold a flat sheet to cover it.

We can use parallel transport to define curvature. If vectors which are transported round a closed loop always return to their starting values, then the surface is flat. If not, we can define the curvature as in the following section.

End of Lecture 10

12 Gravity

12.1 Preamble

How do we tell if we are in a gravitational field? We know we can *remove* gravity locally, by moving to a LIF, $g_{\mu\nu} \to \eta_{\mu\nu}$, in which the affine connections Γ vanish. i.e. $g_{\mu\nu,\lambda} = 0$. The presence of a *'real'* gravitational field shows in the presence of non-zero *second*-derivatives of g. These lead to *tidal forces*: neighbouring points feel different accelerations, because the affine connections differ. Making a Taylor expansion of $g_{\mu\nu,\lambda}$,

$$g_{\mu\nu,\lambda}(x + \Delta x) \simeq g_{\mu\nu,\lambda}(x) + g_{\mu\nu,\lambda\sigma}\Delta x^\sigma \tag{128}$$

Hence the forces at two points separated by Δx^σ will be different if

$$g_{\mu\nu,\lambda\sigma} \neq 0. \tag{129}$$

This is a 4th-rank object which somehow contains information about the presence of a gravitational field. Unfortunately, it is not a tensor, so if one observer thinks it is zero, others may not agree. We have to work a little harder to find a 4th-rank tensor which does the trick. It turns out that the relevant tensor is a tensor which describes the curvature of the space.

End of Lecture 10

12.2 The curvature tensor

Parallel-transport a vector V^α around a small closed parallelogram with sides δa^μ and δb^ν. From (123), the change along a side δx^β is

$$\delta V^\alpha = -\Gamma^\alpha{}_{\beta\nu} V^\nu \delta x^\beta \tag{130}$$

The total change after going round the parallelogram is

$$\begin{aligned}\delta V^\alpha \quad = \quad & -\Gamma^\alpha{}_{\beta\nu}(x) V^\nu(x) \delta a^\beta - \Gamma^\alpha{}_{\beta\nu}(x+\delta a) V^\nu(x+\delta a)\delta b^\beta \\ & + \Gamma^\alpha{}_{\beta\nu}(x+\delta b) V^\nu(x+\delta b)\delta a^\beta + \Gamma^\alpha{}_{\beta\nu}(x) V^\nu(x)\delta b^\beta\end{aligned} \tag{131}$$

We assume the loop is small in comparison with the scale over which Γ changes. Making a Taylor expansion of ΓV, we find

$$\delta V^\alpha = \frac{\partial(\Gamma^\alpha{}_{\beta\nu} V^\nu)}{\partial x^\rho}\delta b^\rho \delta a^\beta - \frac{\partial(\Gamma^\alpha{}_{\beta\nu} V^\nu)}{\partial x^\rho}\delta a^\rho \delta b^\beta \tag{132}$$

Swapping the dummy indices $\rho \leftrightarrow \beta$ in the second term, (so the labels on δa and δb agree with the first term) , and differentiating the products:

$$\delta V^\alpha = \left(\Gamma^\alpha{}_{\beta\nu,\rho} V^\nu + \Gamma^\alpha{}_{\beta\nu} V^\nu{}_{,\rho} - \Gamma^\alpha{}_{\rho\nu,\beta} V^\nu - \Gamma^\alpha{}_{\rho\nu} V^\nu{}_{,\beta}\right) \delta a^\beta \delta b^\rho \tag{133}$$

Now the derivatives of V are (using parallel transport (123) again),

$$V^\nu{}_{,\rho} = -\Gamma^\nu{}_{\sigma\rho} V^\sigma \tag{134}$$

so

$$\delta V^\alpha = \left(\Gamma^\alpha{}_{\beta\nu,\rho} V^\nu - \Gamma^\alpha{}_{\beta\nu}\Gamma^\nu{}_{\sigma\rho} V^\sigma - \Gamma^\alpha{}_{\rho\nu,\beta} V^\nu + \Gamma^\alpha{}_{\rho\nu}\Gamma^\nu{}_{\sigma\beta} V^\sigma\right) \delta a^\beta \delta b^\rho \tag{135}$$

Relabelling dummy indices to make all V terms V^σ, we get finally

$$\delta V^\alpha \equiv R^\alpha{}_{\sigma\rho\beta} V^\sigma \delta a^\beta \delta b^\rho \tag{136}$$

where $R^\alpha{}_{\sigma\rho\beta}$ is the *Riemann curvature tensor*:

$$\boxed{R^\alpha{}_{\sigma\rho\beta} \equiv \Gamma^\alpha{}_{\beta\sigma,\rho} - \Gamma^\alpha{}_{\rho\sigma,\beta} + \Gamma^\alpha{}_{\rho\nu}\Gamma^\nu{}_{\sigma\beta} - \Gamma^\alpha{}_{\beta\nu}\Gamma^\nu{}_{\sigma\rho}} \tag{137}$$

If the Riemann curvature tensor is zero, then any vector parallel-transported round the loop will not change, and the space is flat. If the Riemann tensor is non-zero, vectors do change in general, and we deduce that the space is curved. $R^\alpha{}_{\sigma\beta\rho}$ is a tensor (by its construction from the tensors δV^α, V^σ, δa^β and δb^ρ), so if it is zero in one frame, it is zero in all - i.e. *all* observers agree on whether space is flat ($R^\alpha{}_{\sigma\beta\rho} = 0$) or curved ($R^\alpha{}_{\sigma\beta\rho} \neq 0$).

We have created a tensor which defines the curvature of spacetime. It depends on the second derivative of $g_{\alpha\beta}$, which we suspect to be connected with the presence of tidal forces (i.e. 'real' gravity). This suggests an intimate link between curvature and gravity. We make this explicit in the next section.

12.3 Connection with gravity: geodesic deviation

We can locally transform away gravity; its presence is detected by the way that nearby free particles move apart or together, i.e. by *relative deviations of neighbouring geodesics.* Consider 2 such geodesics, $x^\mu(\tau)$ and $x^\mu(\tau) + y^\mu(\tau)$. For arbitrary τ, we will see how the (small) separation y^μ grows. Let P be the spacetime point at $x^\mu(\tau)$.

We now employ a useful general trick which simplifies the algebra. We first work in a local inertial frame at P; then we find an equation in this frame; finally we write the equation as a tensor equation, and this must be general, hence we find the general solution.

Geodesics obey:

$$\begin{aligned} \ddot{x}^\mu + \Gamma^\mu{}_{\nu\lambda}(x)\dot{x}^\nu\dot{x}^\lambda &= 0 \\ \ddot{x}^\mu + \ddot{y}^\mu + \Gamma^\mu{}_{\nu\lambda}(x+y)(\dot{x}^\nu + \dot{y}^\nu)(\dot{x}^\lambda + \dot{y}^\lambda) &= 0 \end{aligned} \tag{138}$$

Now, in a LIF at P, $\Gamma^\mu{}_{\nu\lambda}(x) = 0$, simplifying the algebra. Until the very end, these equations now only hold in the LIF.

Making a Taylor expansion of the second equation to first order, and subtracting the first gives

$$\ddot{y}^\mu + \frac{\partial \Gamma^\mu{}_{\nu\lambda}}{\partial x^\rho}\dot{x}^\nu\dot{x}^\lambda y^\rho = 0 \tag{139}$$

Now $d^2y^\mu/d\tau^2$ is not a tensor, so different observers will generally disagree on whether it is zero or not. They *will* all agree on whether the covariant relative acceleration is zero, since is a tensor.

The covariant derivative is

$$\begin{aligned} \frac{D^2y^\mu}{D\tau^2} &= \frac{D}{D\tau}\left(\frac{Dy^\mu}{D\tau}\right) = \frac{d}{d\tau}\left[\frac{dy^\mu}{d\tau} + \Gamma^\mu{}_{\lambda\rho}\dot{x}^\lambda y^\rho\right] \text{ (since } \Gamma = 0) \\ &= \frac{d^2y^\mu}{d\tau^2} + \frac{\partial \Gamma^\mu{}_{\lambda\rho}}{\partial x^\nu}\dot{x}^\nu\dot{x}^\lambda y^\rho \end{aligned} \tag{140}$$

Using (139), this simplifies to

$$\frac{D^2y^\mu}{D\tau^2} = \left(\frac{\partial \Gamma^\mu{}_{\lambda\rho}}{\partial x^\nu} - \frac{\partial \Gamma^\mu{}_{\nu\lambda}}{\partial x^\rho}\right)\dot{x}^\nu\dot{x}^\lambda y^\rho \tag{141}$$

Now, the term in brackets is not a tensor, but we see from comparison with the definition of the Riemann curvature tensor (equation refRiemann), that in the LIF, the two are equal. Hence in this frame we can write

$$\frac{D^2 y^\mu}{D\tau^2} = R^\mu{}_{\nu\lambda\rho}\dot{x}^\nu \dot{x}^\lambda y^\rho. \tag{142}$$

Now this is a tensor relation, so is valid in all frames, and this is the final result. Importantly, what it does is to establish the connection between curvature ($R^\mu{}_{\nu\lambda\rho}$) and gravity (through $D^2y^\mu/D\tau^2$). The meaning of the equation is as follows. In flat space, the Riemann tensor is zero, and we can use cartesian coordinates. The solution is that y^μ grows (or reduces) linearly with τ (since $\Gamma = 0$ and the covariant derivative is just $\ddot{y}^\mu$, which is zero). If the two paths are parallel initially, then they stay parallel. In curved space, though, geodesics which start off parallel may not remain so, because of the non-zero r.h.s. As a 2D example, two close parallel paths heading North from the equator will meet, at the North Pole.

End of Lecture 11

13 Calculations II: The Riemann Tensor

Christoffel symbols $\Gamma^\alpha{}_{\beta\gamma}$

Construct matrices Γ_β, with components (row=α, column=γ)

$$(\Gamma_\beta)^\alpha{}_\gamma \equiv \Gamma^\alpha{}_{\beta\gamma} \tag{143}$$

Riemann tensor

Construct 16 4×4 matrices, labelled by ρ and σ, $B_{\rho\sigma}$, with components (row=α, column=γ)

$$(B_{\rho\sigma})^\alpha{}_\gamma \equiv R^\alpha{}_{\gamma\rho\sigma} \tag{144}$$

Hence B are defined by the matrix equation

$$B_{\rho\sigma} = \partial_\rho\Gamma_\sigma - \partial_\sigma\Gamma_\rho + \Gamma_\rho\Gamma_\sigma - \Gamma_\sigma\Gamma_\rho \tag{145}$$

and are clearly antisymmetric in ρ and σ $\Rightarrow$ we need to compute only 6 B matrices. We can then read off the Riemann tensor components from the elements of $B_{\rho\sigma}$. (Note the distinction between the *labels* ρ and σ, and the rows and columns α and γ).

Symmetries of R (without proof)

$$\begin{aligned} R^\alpha{}_{\mu\rho\sigma} &= -R^\alpha{}_{\mu\sigma\rho} \\ R^\alpha{}_{\rho\sigma\mu} + R^\alpha{}_{\sigma\mu\rho} + R^\alpha{}_{\mu\rho\sigma} &= 0 \\ R_{\alpha\mu\rho\sigma} &= -R_{\mu\alpha\rho\sigma} \\ R_{\alpha\mu\rho\sigma} &= R_{\rho\sigma\alpha\mu} \end{aligned} \tag{146}$$

Contractions:

$$\begin{aligned} R_{\alpha\beta} &\equiv R^\mu{}_{\alpha\mu\beta} && \text{Ricci Tensor (symmetric)} \\ R &\equiv R^\alpha{}_\alpha && \text{Ricci scalar} \end{aligned} \tag{147}$$

14 Einstein's Field Equations

In Newtonian gravity, there is a *field equation* which relates the gravitational *potential* φ to the distribution of matter ρ. It is Poisson's equation

$$\nabla^2 \varphi = 4\pi G\rho \tag{148}$$

We argued in section 3 that the components of the metric tensor $g_{\mu\nu}$ play the role of potentials, so we seek a field equation for them, which satisfies the Principle of General Covariance, and the Correspondence Principle. We place an additional constraint (a guess...), motivated partly by Poisson's equation, that the field equation is linear in g and its first two derivatives, at most. It turns out (no proof) that the only suitable tensor is the Riemann curvature tensor.

14.1 Equations in empty space

Analogue of Laplace's equation $\nabla^2\varphi = 0$.

Neighbouring particles experience a tidal acceleration (142)

$$\begin{aligned} \frac{D^2 y^\mu}{D\tau^2} &= R^\mu{}_{\nu\lambda\rho}\dot{x}^\nu \dot{x}^\lambda y^\rho \\ &\equiv \Delta^\mu{}_\rho y^\rho \end{aligned} \tag{149}$$

where $\Delta^\mu{}_\rho$ is the *tidal tensor*. The 3D tidal tensor in Newtonian gravity is

$$\Delta_{ij}^{Newtonian} = -\frac{\partial^2 \varphi}{\partial x^i \partial x^j} \tag{150}$$

for $i, j = 1, 2, 3$ (look at difference in i acceleration of points separated by δx^j). Laplace's equation is then

$$\Delta_{ii}^{Newtonian} = 0 \tag{151}$$

A generally-covariant generalisation of this is

$$\Delta^\mu{}_\mu = 0 \qquad \text{in empty space.} \tag{152}$$

i.e.

$$\begin{aligned} R^\mu{}_{\alpha\mu\beta}\dot{x}^\alpha \dot{x}^\beta &= 0 \\ \Rightarrow R_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta &= 0 \end{aligned} \tag{153}$$

To be true for all $\dot{x}^\alpha$, we need

$$\boxed{R_{\alpha\beta} = 0 \qquad \text{in empty space.}} \tag{154}$$

The only other possibility (justified later) is

$$\boxed{R_{\alpha\beta} = \Lambda g_{\alpha\beta} \qquad \text{in empty space.}} \tag{155}$$

where Λ is the famous *Cosmological Constant*, introduced by Einstein, since, without it, a static Universe is not possible. Einstein developed this before Slipher and Hubble discovered that the Universe was expanding. Λ introduces a repulsive force $\propto r$, which can be negligible in the Solar System, but important on cosmological scales.

15 Einstein's equations with matter

15.1 The source of gravity: the energy-momentum tensor

In Newtonian gravity, ρ is the source of ϕ ($\nabla^2\varphi = 4\pi G\rho$). In GR, we seek a tensor equation relating the potentials ($g_{\mu\nu}$) to the matter. It cannot be a scalar equation, because ρ is not a scalar. We can get some insight from SR into what rank of tensor equation it has to be, because, if ρ_0 is the density in the rest frame of a fluid, then the density measured by an observer moving with Lorentz factor γ will be

$$\rho = \gamma^2\rho_0 \tag{156}$$

where one factor comes from Lorentz contraction, and another from the relativistic increase in mass of the fluid particles. Transforming second-rank tensors in SR brings in 2 factors of γ.

End of Lecture 12

The simplest tensor we can make which has a component which reduces to ρc^2 in the Newtonian limit is

$$T^{\mu\nu} = \rho_0 U^\mu U^\nu \tag{157}$$

where $U^\mu = \gamma(c, \mathbf{u})$ is the 4-velocity and from now on we drop the 0 subscript and ρ refers to the rest-frame density. This is the *energy-momentum tensor* of 'dust' (technical term: pressure-free and viscosity-free fluid). The equations of mass and momentum conservation are contained in the 4-divergence of this tensor:

$$T^{\mu\nu}{}_{,\nu} = \nabla_\nu T^{\mu\nu} = 0 \tag{158}$$

In the limit $\gamma = 1$, $\mu = 0$ unpacks to the continuity equation

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0 \tag{159}$$

and the spatial parts give

$$\frac{\partial}{\partial t}(\rho u_i) + \nabla_k(\rho u_i u_k) = 0 \tag{160}$$

which, with the continuity equation can be manipulated into Euler's equation $du_i/dt = (\partial u_i/\partial t) + (\mathbf{u}\cdot\nabla)u_i = 0$ for a pressure-free fluid. For a fluid with pressure, we construct the energy-momentum tensor (or *stress-energy tensor*)

$$T^{\mu\nu} = \left(\rho + \frac{p}{c^2}\right)U^\mu U^\nu - p\eta^{\mu\nu} \tag{161}$$

The conservation law

$$T^{\mu\nu}{}_{,\nu} = 0 \tag{162}$$

incorporates conservation of energy and momentum. Note that ρ and p are defined in the rest frame of the fluid so are scalars.

For GR, we follow the algorithm of section 11. i.e. $\{, \rightarrow ;\}$

$$\boxed{T^{\mu\nu}{}_{;\nu} = 0} \tag{163}$$

and $\eta \rightarrow g$:

$$\boxed{T^{\mu\nu} = \left(\rho + \frac{p}{c^2}\right)U^\mu U^\nu - pg^{\mu\nu}} \tag{164}$$

Note that the conservation law may be expressed as

$$\frac{\partial T^{\mu\nu}}{\partial x^\nu} + \Gamma^\mu{}_{\alpha\nu}T^{\alpha\nu} + \Gamma^\nu{}_{\alpha\nu}T^{\alpha\mu} = 0 \tag{165}$$

and the Γ terms represent kinematic (or *inertial*) forces. In their absence, we have a conservation law. Note that, for short times, or short distances ($\ll$ scales over which curvature changes) we can work in a locally inertial frame, in which the Γ are (instantaneously, at least) zero. This leads to an approximate *local* conservation law.

A final note. Electromagnetism is a *vector* theory, because charge is experimentally seen to be conserved. Thus charge density ρ_Q is proportional to γ, as only Lorentz contraction matters. This allows a vector source ($J^\mu = (c\rho_Q, \mathbf{j})$) in the equation for the EM 4-potential. The dependence of mass on velocity in GR introduces a second γ and thus GR is a second-rank tensor theory.

15.2 Einstein Field Equations

We have generalised the source term ρ; now for $\nabla^2\varphi$. We have a second-rank tensor for the source term, so we seek a second-rank tensor involving derivatives of $g_{\mu\nu}$. An obvious candidate is the Ricci tensor:

$$R^{\alpha\beta} = \text{constant } T^{\alpha\beta} \qquad \text{(wrong)} \tag{166}$$

but this fails because it turns out that $R^{\alpha\beta}{}_{;\beta} \neq 0$. There is, however, a tensor whose covariant divergence is always zero. It is the *Einstein tensor*:

$$G^{\mu\nu} \equiv R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R \tag{167}$$

and $G^{\mu\nu}{}_{;\nu} = 0$. (See Weinberg p. 146-7). Compelling to postulate that:

$$G^{\mu\nu} = aT^{\mu\nu} \tag{168}$$

where a is a constant. Solving for the constant using the Correspondence Principle (next section) gives

$$\boxed{G^{\mu\nu} = \frac{8\pi G}{c^4}T^{\mu\nu}} \tag{169}$$

15.2.1 The cosmological constant Λ

The field equations (169) do not admit static solutions for the Universe (see later). Einstein could therefore have predicted an expanding (or contracting) Universe, but instead chose to add a term to the equations to permit a static solution. We noted earlier that $g^{\mu\nu}{}_{;\nu} = 0$, so we can add any multiple Λ of $g^{\mu\nu}$ to $G^{\mu\nu}$ and still get a quantity whose covariant divergence vanishes. Λ is the *cosmological constant*; it cannot be too big, to avoid disturbing Newtonian gravity, but it corresponds to a force which is proportional to r, so can be negligible in the Solar System but important cosmologically.

$$\boxed{G^{\mu\nu} - \Lambda g^{\mu\nu} = \frac{8\pi G}{c^4}T^{\mu\nu}} \tag{170}$$

15.3 Determining the constant a

Choose simplest non-trivial situation: time-independent, weak-field, low speed. Write

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{171}$$

with $|h_{\mu\nu}| \ll 1$. Energy-momentum tensor is

$$T_{00} = \rho c^2; \qquad T_{ij} \simeq 0 \ (\ll T_{00}) \tag{172}$$

To $O(h)$, the affine connections are

$$\begin{aligned} \Gamma^\sigma{}_{\lambda\mu} &= \frac{1}{2}g^{\nu\sigma}\left\{g_{\mu\nu,\lambda} + g_{\lambda\nu,\mu} - g_{\mu\lambda,\nu}\right\} \\ &\simeq \frac{1}{2}\eta^{\nu\sigma}\left\{h_{\mu\nu,\lambda} + h_{\lambda\nu,\mu} - h_{\mu\lambda,\nu}\right\} \end{aligned} \tag{173}$$

The Riemann tensor is (137)

$$R^{\alpha}{}_{\sigma\rho\beta} \equiv \Gamma^{\alpha}{}_{\beta\sigma,\rho} - \Gamma^{\alpha}{}_{\rho\sigma,\beta} + \Gamma^{\alpha}{}_{\rho\nu}\Gamma^{\nu}{}_{\sigma\beta} - \Gamma^{\alpha}{}_{\beta\nu}\Gamma^{\nu}{}_{\sigma\rho} \tag{174}$$

so to $O(h)$ we ignore the $\Gamma\Gamma$ terms, and

$$R^{\alpha}{}_{\sigma\rho\beta} \simeq \frac{1}{2}\eta^{\alpha\nu}\left\{h_{\beta\nu,\sigma\rho} - h_{\beta\sigma,\nu\rho} - h_{\rho\nu,\sigma\beta} + h_{\rho\sigma,\beta\nu}\right\} \tag{175}$$

(two h terms cancel). The Ricci tensor is then

$$R_{\sigma\beta} = R^{\alpha}{}_{\sigma\alpha\beta} \simeq \frac{1}{2}\eta^{\alpha\nu}\left\{h_{\beta\nu,\sigma\alpha} - h_{\beta\sigma,\nu\alpha} - h_{\alpha\nu,\sigma\beta} + h_{\alpha\sigma,\beta\nu}\right\} \tag{176}$$

We will use only $G_{00} = R_{00} - (1/2)g_{00}R$ to work out the constant of proportionality, so look here at R_{00}. For a static field, $\partial/\partial t = 0$, so derivatives w.r.t. β and σ vanish, leaving only the second term in the bracket:

$$\begin{aligned} R_{00} &\simeq \frac{1}{2}\eta^{\alpha\nu}\left\{-h_{00,\nu\alpha}\right\} \\ &\simeq \frac{1}{2}\nabla^2 h_{00} \end{aligned} \tag{177}$$

since h_{00} has no time derivative. Previously (35) we had $g_{00} \simeq 1 + 2\varphi/c^2$, so

$$R_{00} = \frac{1}{c^2}\nabla^2\varphi \tag{178}$$

to $O(h)$.

To get R, note that, since $|T_{ij}| \to 0$ in the non-relativistic limit, $|G_{ij}| \to 0$, or

$$\begin{aligned} R_{ij} - \frac{1}{2}g_{ij}R &\simeq 0 \\ \Rightarrow R_{ij} &\simeq \frac{1}{2}\eta_{ij}R \simeq -\frac{1}{2}\delta_{ij}R \end{aligned} \tag{179}$$

and the Ricci scalar is

$$R = R^{\mu}{}_{\mu} \simeq \eta^{\mu\nu}R_{\nu\mu} = R_{00} - R_{ii} = R_{00} + \frac{3}{2}R \tag{180}$$

using (179) for R_{ii}. Hence $R \simeq -2R_{00}$ and $G_{00} = R_{00} - \frac{1}{2}g_{00}R \simeq 2R_{00}$. Hence $G_{00} = aT_{00} \Rightarrow$

$$\frac{2}{c^2}\nabla^2\varphi = a\rho c^2 \tag{181}$$

Since this must reduce to Poisson's equation $\nabla^2\varphi = 4\pi G\rho$, the constant has to be $a = 8\pi G/c^4$.

End of Lecture 13

16 Cosmology

16.1 The Robertson-Walker metric

As with the Schwarzschild metric, we can constrain the *form* of the metric on symmetry grounds, before using the field equations to find the solution.

Symmetry: Universe is *homogeneous* and *isotropic* (i.e. is the same everywhere, and looks the same in all directions; note these are not logically identical). Evidence of isotropy – microwave background temperature, after Earth's motion subtracted.

Expansion: Hubble's law – galaxies recede with $v \propto r$.

We assume a uniform universe with no deviations from this expansion law, and no deviations in density from place-to-place.

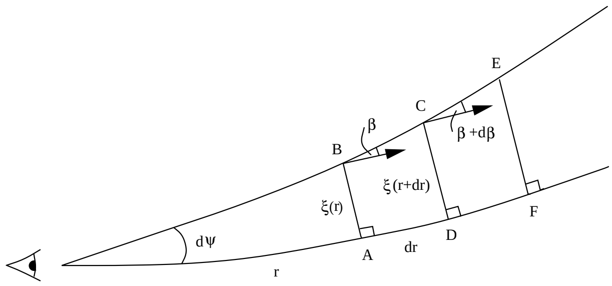

Figure 4: Scheuer's argument for $f(r)$. Note: figure has different labels - for r read ρ, for ξ read Z, for $d\Psi$ read $\Delta\theta$.

16.1.1 Choice of coordinates

1) Time t. Take to be time elapsed on clocks for which the microwave background is isotropic. Note this is not a *preferred frame* against the principles of relativity. Because of expansion, such clocks move apart from one another.

2) Spatial coordinates: choose spherical polars r, θ, ϕ, and choose them to be *comoving* i.e. each galaxy retains its r label, even though it is moving away. Actual distances are $R(t)r$, where $R(t)$ is the *Cosmic Scale Factor*, which is independent of position, by homogeneity. Choose r to be an *angular diameter distance*, so the comoving spatial separation of points is $d\ell^2 = f(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$. The Robertson-Walker metric then has the form

$$ds^2 = c^2dt^2 - R^2(t)\left[f(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] \tag{182}$$

No GR yet – only symmetry.

16.2 Peter Scheuer's argument for $f(r)$

See Longair, Theoretical Concepts in Physics.

r is an angular diameter coordinate. If we call the ruler, or geodesic, coordinate ρ, then inspection of the metric shows us that

$$d\rho = \sqrt{f(r)}dr \tag{183}$$

Parallel-transport a vector round ABC, and ADC, where each arm of the quadrilateral is a geodesic. In the first case, the angle between the vector and the geodesic becomes β at C; in the latter it is different, in general, i.e. $\beta + d\beta$. Clearly

$$\beta = \frac{dZ}{d\rho} \tag{184}$$

Note that ρ is a geodesic distance; it is not the angular diameter distance r - with this labelling, the distance AD is $d\rho$. Hence

$$\beta + d\beta = \frac{dZ}{d\rho} + \frac{d^2Z}{d\rho^2}d\rho \tag{185}$$

For a homogeneous and isotropic space, $d\beta$ can depend only on the area of the loop (not its orientation or shape), $d\beta \propto Zd\rho$. Thus

$$d\beta = \frac{d^2Z}{d\rho^2}d\rho = -KZd\rho \tag{186}$$

for some constant K. Hence, assuming $K > 0$ for now

$$Z = Z_0 \sin\left(\sqrt{K}\rho\right) \tag{187}$$

($\sinh(\sqrt{-K}\rho)$ if $K < 0$). For small r, space is almost flat, so $Z \to \rho\Delta\theta$, so $Z_0 = \Delta\theta/\sqrt{K}$, and

$$Z = \frac{\Delta\theta}{\sqrt{K}} \sin\left(\sqrt{K}\rho\right) \tag{188}$$

and similarly for $K < 0$. For $K = 0$, $Z = \rho\Delta\theta$.

Since r is defined so that $Z \equiv r\Delta\theta$, we see that

$$r = \frac{1}{\sqrt{K}} \sin(\sqrt{K}\rho) \tag{189}$$

for $K > 0$, so $dr = \cos(\sqrt{K}\rho)d\rho$, and therefore

$$d\rho = \frac{dr}{\sqrt{1 - Kr^2}} \tag{190}$$

Similarly, with $K < 0$, we take $dr = \cosh(\sqrt{-K}\rho)d\rho$, and we get the *same* expression for $d\rho$, which also obviously holds for flat space ($K = 0$; $r = \rho$). For all 3 cases, we may write

$$\boxed{ds^2 = c^2dt^2 - R^2(t)\left[\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right]} \tag{191}$$

and we can choose length units such that $k = 0, \pm 1$. Still no GR, which is needed now to compute k and $R(t)$.

Alternative is (relabelling ρ to the more common r):

$$\boxed{ds^2 = c^2dt^2 - R^2(t)\left[dr^2 + S_k(r)^2(d\theta^2 + \sin^2\theta d\phi^2)\right]} \tag{192}$$

where $S_k(r) = \sin(r), \sinh(r)$ or r.

From the Euler-Lagrange equations, we get the Christoffel symbols, and write them in matrix form. Writing $a \equiv 1 - kr^2$,

$$\Gamma_t = \begin{pmatrix} . & . & . & . \\ . & R'/R & . & . \\ . & . & R'/R & . \\ . & . & . & R'/R \end{pmatrix} \qquad \Gamma_r = \begin{pmatrix} . & RR'/a & . & . \\ R'/R & kr/a & . & . \\ . & . & 1/r & . \\ . & . & . & 1/r \end{pmatrix} \tag{193}$$

$$\Gamma_\theta = \begin{pmatrix} . & . & R'Rr^2 & . \\ . & . & -ar & . \\ R'/R & 1/r & . & . \\ . & . & . & \cot\theta \end{pmatrix} \qquad \Gamma_\phi = \begin{pmatrix} . & . & . & RR'r^2\sin^2\theta \\ . & . & . & -ar\sin^2\theta \\ . & . & . & -\sin\theta\cos\theta \\ R'/R & 1/r & \cot\theta & . \end{pmatrix} \tag{194}$$

where $' \equiv d/dt$.

The 6 independent $B_{\rho\sigma}$ matrices, from

$$B_{\rho\sigma} = \partial_\rho \Gamma_\sigma - \partial_\sigma \Gamma_\rho + \Gamma_\rho \Gamma_\sigma - \Gamma_\sigma \Gamma_\rho \tag{195}$$

are,

$$B_{tr} = \begin{pmatrix} . & R''R/a & . & . \\ R''/R & . & . & . \\ . & . & . & . \\ . & . & . & . \end{pmatrix} \qquad B_{t\theta} = \begin{pmatrix} . & . & R''Rr^2 & . \\ . & . & . & . \\ R''/R & . & . & . \\ . & . & . & . \end{pmatrix} \tag{196}$$

$$B_{t\phi} = \begin{pmatrix} . & . & . & R''Rr^2\sin^2\theta \\ . & . & . & . \\ . & . & . & . \\ R''/R & . & . & . \end{pmatrix}; \qquad B_{r\theta} = \begin{pmatrix} . & . & . & . \\ . & . & r^2(R'^2+k) & . \\ . & -(R'^2+k)/a & . & . \\ . & . & . & . \end{pmatrix} \tag{197}$$

$$B_{r\phi} = \begin{pmatrix} . & . & . & . \\ . & . & . & (R'^2+k)\sin^2\theta \\ . & . & . & . \\ . & -(R'^2+k)/a & . & . \end{pmatrix} \tag{198}$$

$$B_{\theta\phi} = \begin{pmatrix} . & . & . & . \\ . & . & . & . \\ . & . & . & (R'^2+k)r^2\sin^2\theta \\ . & . & -(R'^2+K)r^2 & . \end{pmatrix} \tag{199}$$

To get the Ricci tensor, first find the non-zero Riemann tensor elements $R^\alpha_{\beta\rho\sigma}$ = Row α, column β of $B_{\rho\sigma}$:

$$\begin{aligned} R^t{}_{rtr} &= R''R/a \\ R^r{}_{ttr} &= R''/R \\ R^t{}_{\theta t\theta} &= R''Rr^2 \\ R^\theta{}_{tt\theta} &= R''/R \\ R^t{}_{\phi t\phi} &= R''Rr^2\sin^2\theta \\ R^\phi{}_{tt\phi} &= R''/R \\ R^r{}_{\theta r\theta} &= (R'^2+k)r^2 \\ R^\theta{}_{rr\theta} &= -(R'^2+k)/a \\ R^r{}_{\phi r\phi} &= (R'^2+k)r^2\sin^2\theta \\ R^\phi{}_{rr\phi} &= -(R'^2+k)/a \\ R^\theta{}_{\phi\theta\phi} &= (R'^2+k)r^2\sin^2\theta \\ R^\phi{}_{\theta\theta\phi} &= -(R'^2+k)r^2 \end{aligned} \tag{200}$$

To get the Ricci tensor, $R_{\alpha\beta} = R^\mu{}_{\alpha\mu\beta}$, we need to make use of the (anti-)symmetry of the Riemann tensor (from anti-symmetry of $B_{\rho\sigma}$:

$$R^\alpha{}_{\beta\rho\sigma} = -R^\alpha{}_{\beta\sigma\rho} \tag{201}$$

(so, for example, $R^\phi{}_{t\phi t} = -R^\phi{}_{tt\phi} = R''/R$).

$$R_{\mu\nu} = \begin{pmatrix} -3R''/R & . & . & . \\ . & A/a & . & . \\ . & . & Ar^2 & . \\ . & . & . & Ar^2\sin^2\theta \end{pmatrix} \tag{202}$$

where $A \equiv RR'' + 2(R'^2 + k)$, and the Ricci scalar is

$$R_s = -6\left(\frac{R''}{R} + \frac{R'^2 + k}{R^2}\right) \tag{203}$$

and the (contravariant) Einstein tensor is (top-left quarter only)

$$G^{\mu\nu} = \begin{pmatrix} 3(R'^2+k)/R^2 & . & . & . \\ . & a\left[-2R''/R - (R'^2+k)/R^2\right] & . & . \\ . & . & \dots & . \\ . & . & . & \dots \end{pmatrix} \tag{204}$$

The Energy-momentum tensor is $T^{\mu\nu} = (\rho + p)U^\mu U^\nu - pg^{\mu\nu}$, and $U^\mu = (1, \mathbf{0})$, so

$$T^{\mu\nu} = \begin{pmatrix} \rho & . & . & . \\ . & ap/R^2 & . & . \\ . & . & p/(R^2r^2) & . \\ . & . & . & p/(R^2r^2\sin^2\theta) \end{pmatrix} \tag{205}$$

and the Einstein field equations $G^{\mu\nu} - \Lambda g^{\mu\nu} = \frac{8\pi G}{c^4}T^{\mu\nu}$ give

$$\boxed{\begin{aligned} R'^2 + kc^2 - \frac{\Lambda}{3}c^2R^2 &= \frac{8\pi G\rho}{3}R^2 \\ 2\frac{R''}{R} + \frac{R'^2 + kc^2}{R^2} - \Lambda c^2 &= -\frac{8\pi Gp}{c^2} \end{aligned}} \tag{206}$$

These are the fundamental equations which govern the expansion of the Universe.

<u>End of Lecture</u> <u>14</u>

17 Gravitational Waves

For weak fields, from (175) the Riemann tensor is

$$R_{\rho\beta\mu\nu} \simeq \frac{1}{2}\left\{h_{\rho\nu,\beta\mu} - h_{\beta\nu,\rho\mu} - h_{\rho\mu,\nu\beta} + h_{\beta\mu,\nu\rho}\right\} \tag{207}$$

Consider plane waves with wavevector $K^\mu = (\omega/c, \mathbf{k})$, so

$$h_{\alpha\beta} = a_{\alpha\beta}\exp(iK_\mu x^\mu) \tag{208}$$

for some constants $a_{\mu\nu}$. Note that, in weak fields, $K_\mu \simeq \eta_{\mu\nu}K^\nu = (\omega/c, -\mathbf{k})$, so $K_\mu x^\mu = \omega t - \mathbf{k}\cdot\mathbf{r}$. Differentiating: $\partial_\rho \to iK_\rho$, so

$$R_{\rho\beta\mu\nu} \simeq \frac{1}{2}\left\{-K_\nu K_\rho h_{\beta\mu} + K_\nu K_\beta h_{\rho\mu} + K_\rho K_\mu h_{\beta\nu} - K_\beta K_\mu h_{\rho\nu}\right\} \tag{209}$$

and the Ricci tensor is

$$R_{\beta\nu} = R^\rho{}_{\beta\rho\nu} = \frac{1}{2}\left\{-K_\nu K^\rho h_{\beta\rho} + K_\nu K_\beta h^\rho{}_\rho + K^\rho K_\rho h_{\beta\nu} - K_\beta K_\rho h^\rho{}_\nu\right\} \tag{210}$$

Raising one index in the first term and lowering another, and defining $K^2 \equiv K^\rho K_\rho$,

$$\begin{aligned} R_{\beta\nu} &= \frac{1}{2}\left\{-K_\nu K_\rho h^\rho{}_\beta + K_\beta K_\nu h^\rho{}_\rho + K^2 h_{\beta\nu} - K_\beta K_\rho h^\rho{}_\nu\right\} \\ &= \frac{1}{2}\left\{K^2 h_{\beta\nu} - K_\nu\omega_\beta - K_\beta\omega_\nu\right\} \end{aligned} \tag{211}$$

where

$$\omega_\nu \equiv K_\rho h^\rho{}_\nu - \frac{1}{2}K_\nu h^\rho{}_\rho. \tag{212}$$

Hence, for a wave propagating through empty space, where $R_{\beta\nu} = 0$,

$$K^2 h_{\beta\nu} = K_\nu\omega_\beta + K_\beta\omega_\nu. \tag{213}$$

Substituting for $h_{\beta\nu}$ from (213) in (209) gives

$$K^2 R_{\rho\beta\mu\nu} = 0 \tag{214}$$

(all the terms on the r.h.s. cancel out). There are two solutions: if $K^2 \neq 0$, then the Riemann tensor vanishes. *This is not a true wave at all*, since no curvature is induced. It is simply a sinusoidal change in the coordinate system. There are four degrees of freedom to do this - each new coordinate can be an arbitrary function of the old coordinates.

We get real waves if $K^2 = 0$, for which $K_\nu \omega_\beta + K_\beta \omega_\nu = 0$, which requires $\omega_\nu = 0$ (e.g. take $\beta = \nu = 0$; $K_0\omega_0 = 0 \Rightarrow \omega_0 = 0$ since $K_0 \neq 0$ in general. Similarly for other components). From (212), we see that the waves satisfy the 4 conditions

$$K_\rho h^\rho{}_\nu = \frac{1}{2} K_\nu h^\rho{}_\rho. \tag{215}$$

The Riemann tensor does not vanish in this case, but we will need the fact that it satisfies

$$R_{\rho\beta\mu\nu} K^\nu = 0 \tag{216}$$

(Exercise: prove this. Hints: $K^\nu K_\nu = K^2 = 0$, and you will need (215), but you will need to raise and lower some indices to use it).

$h_{\mu\nu}$ has 10 independent components, since it is symmetric. The 4 conditions (215) reduce this to 6, and an arbitrary coordinate transformation (cf the case $K^2 \neq 0$ above) transformation reduces this to 2. Thus there are 2 remaining degrees of freedom – 2 polarizations of transverse waves.

Note that $K^2 = 0 \Rightarrow \omega^2/c^2 - \mathbf{k}^2 = 0$, so gravitational waves travel at the speed of light (or light travels at the speed of gravity).

17.1 What does a gravity wave do?

Gravity waves induce tidal forces. Tidal tensor is $\Delta_{\mu\sigma} = R_{\mu\nu\rho\sigma}\dot{x}^\nu \dot{x}^\rho$ where $\dot{x}^\rho$ is the observer's velocity. As before, it is traceless, $\Delta^\mu{}_\mu = 0$, and $\Delta_{\mu\sigma}\dot{x}^\sigma = 0$ (from symmetries of Riemann tensor - second line of (146)[2]. For a plane, transverse wave, from (216),

$$\Delta_{\mu\sigma} K^\sigma = 0 \tag{217}$$

Example: stationary observer, with $x^\mu = (ct, \mathbf{x}) = (ct, \mathbf{0})$, and gravity wave propagating along the z axis: $K^\mu = (\omega/c, 0, 0, \omega/c)$. Then the 0 and 3 rows and columns of Δ must be zero (to satisfy (217):

$$\Delta_{\mu\sigma} = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & a & b & \cdot \\ \cdot & b & -a & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \times \exp\left[\frac{i\omega}{c}(ct - z)\right] \tag{218}$$

(The form is determined by the requirement that Δ be traceless and symmetric). The two polarisations correspond to $a = 0$ and $b = 0$. Consider effect on a circle at $z = 0$, and initially $b = 0$.

The tidal 3-acceleration is

$$\Delta^i{}_j x^j = \begin{pmatrix} a & \cdot & \cdot \\ \cdot & -a & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \exp(i\omega t) \tag{219}$$

$$\Delta^i{}_j x^j = a \begin{pmatrix} x \\ -y \\ 0 \end{pmatrix} \cos(\omega t) \tag{220}$$

[2]Raise the first index, and relabel dummy indices, to write the term as the sum of 3, with the lower indices of R cyclically permuted: $R^\mu{}_{\nu\rho\sigma}\dot{x}^\nu\dot{x}^\rho\dot{x}^\sigma = (R^\mu{}_{\nu\rho\sigma} + R^\mu{}_{\rho\sigma\nu} + R^\mu{}_{\sigma\nu\rho})\dot{x}^\nu\dot{x}^\rho\dot{x}^\sigma/3 = 0$ by (146).

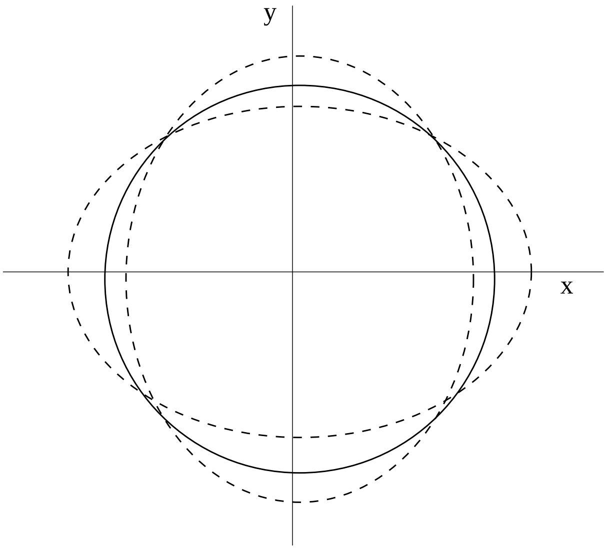

Figure 5: Effect of gravitational wave with + polarisation.

Similarly, the other polarisation, $a = 0$, gives

$$\Delta^i{}_j x^j = b \begin{pmatrix} y \\ x \\ 0 \end{pmatrix} \cos(\omega t) \tag{221}$$

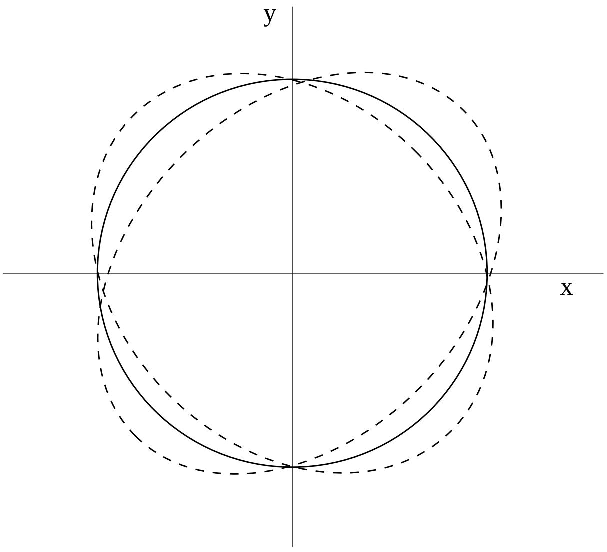

Figure 6: Effect of gravitational wave with $\times$ polarisation.

The axes of the polarizations are at an angle $\pi/4$ to each other, and the wave is a simultaneous squashing and stretching, along orthogonal axes, preserving volume. Note that the effect is a *strain*: the distortion is proportional to size, motivating large (up to 3 km) detectors on Earth (e.g. UK/German GEO-600 detector in Hannover, US LIGO, French/Italian VIRGO detectors), or large space missions (LISA; not yet funded). Detectable strains are in the region $\delta x/x \sim 10^{-19} - 10^{-23}$ for the different experiments, requiring sophisticated optics (work out δx!).

More on gravitational waves at `http://www.lisa.uni-hannover.de/`

End of Lecture 15

18 Black Holes

The Schwarzschild metric (59)

$$ds^2 = \left(1 - \frac{2GM}{rc^2}\right) c^2 dt^2 - \frac{dr^2}{1 - \frac{2GM}{rc^2}} - r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{222}$$

is valid for all r outside a mass M. If the mass is contained within the *Schwarzschild radius*

$$r_s \equiv \frac{2GM}{c^2} \tag{223}$$

then there is clearly something odd at $r = r_s$, since $g_{tt} \to 0$ and $g_{rr} \to \infty$. In fact the metric is singular at 3 'places': $r = 0$, $r = r_s$ and $r \to \infty$. How should we interpret these singularities?

$r \to \infty$ is not a problem. Writing the Minkowski metric in spherical polars gives $ds^2 = c^2dt^2 - dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$, we see that two metric coefficients diverge as $r \to \infty$, but these singularities can be removed by using a cartesian coordinate system. $r \to \infty$ is thus a *coordinate singularity* Similarly, the Schwarzschild metric has only a coordinate singularity at $r \to \infty$. (In flat space, we can create a coordinate singularity at finite distance, by taking, e.g. $X = x^3/3$, in which case, $dx^2 = (3X)^{-4/3}dX^2$, and the metric in terms of X becomes singular at $X = 0$). We clearly need to look closely at singularities to see if they 'really are' singular.

18.1 The singularity at r_s

18.1.1 Event horizon

The *event horizon* marks the boundary of events which can ever be detected in the future. In general cases, it can be difficult to compute, because of having to consider all possible geodesics, but in the very simple metric considered here, it boils down to

$$g_{rr} \to \infty, \tag{224}$$

so that for finite time intervals as measured by a local observer, $dr \to 0$. ($ds^2 = g_{tt}c^2dt^2 - g_{rr}dr^2 = 0$, and $g_{tt}dt^2 = dt^2_{local}$).

Consider radial null ($ds^2 = 0$) geodesics:

$$L^2 = \left(1 - \frac{2GM}{rc^2}\right) c^2\dot{t}^2 - \frac{\dot{r}^2}{1 - \frac{2GM}{rc^2}} \tag{225}$$

Using the ELII equations (48), we get

$$\begin{aligned} \frac{d}{dp}\left[\left(1 - \frac{2GM}{rc^2}\right) c\dot{t}\right] &= 0 \\ \left(1 - \frac{2GM}{rc^2}\right) c\dot{t} &= k = \text{constant} \end{aligned} \tag{226}$$

Substituting into (225) we get $\dot{r}^2 = k^2$ so

$$\dot{r} = \pm k \tag{227}$$

and hence

$$c\frac{dt}{dr} = \frac{c\dot{t}}{\dot{r}} = \pm\frac{r}{r - r_s} \tag{228}$$

with 2 solutions

$$\begin{aligned} ct &= r + r_s \ln|r - r_s| + \text{constant} \\ ct &= -(r + r_s \ln|r - r_s| + \text{constant}) \end{aligned} \tag{229}$$

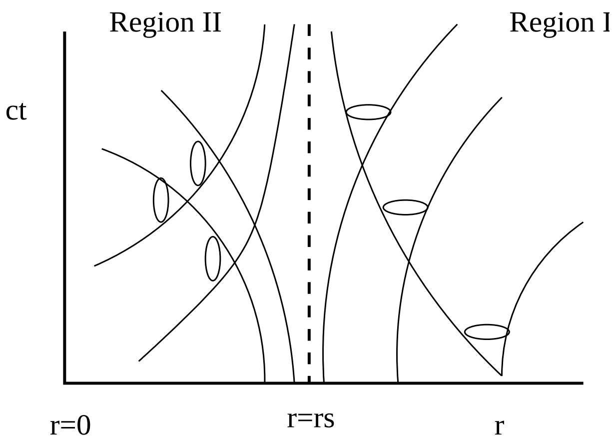

Figure 7: Geodesics and future light cones, near the Schwarzschild radius.

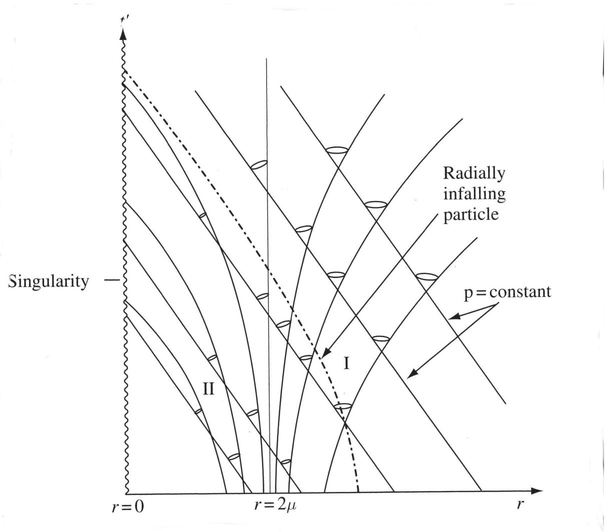

Figure 8: Geodesics and future light cones, near the Schwarzschild radius, in Advanced Eddington-Finkelstein coordinates.

For $r > r_s$, the first solution is an outgoing wave, the second ingoing. In a spacetime diagram, we find geodesics like those shown in the figure. Inside r_s, the future light cones tip over and point towards $r = 0$. All light signals, and hence all matter particles, *must* move towards $r = 0$, which cannot be avoided. Light signals (nor anything else) can get out, so $r = r_s$ is called an *event horizon.*

We also see that, in terms of *coordinate time* t, light signals in region I (outside r_s) take an infinite time to reach r_s.

It is interesting to look at the two solutions inside r_s. For one of them, *ct decreases* as r decreases (see diagram), so *ct* is not a very useful coordinate. It is more useful to define a time coordinate by

$$ct' = ct + r_s \ln |r - r_s| \tag{230}$$

so the null geodesics are

$$\begin{aligned} ct' &= -r + \text{constant} \\ ct' &= r + 2r_s \ln |r - r_s| + \text{constant}. \end{aligned} \tag{231}$$

The resulting coordinates are called *Advanced Eddington-Finkelstein coordinates.* In this system, the light cones look more sensible, and the apparent singularity at $r = r_s$ disappears.

18.1.2 Infinite redshift surface

$g_{tt} = 0$

For a stationary emitter at r, the proper time interval $d\tau$ is related to the coordinate time interval dt, by

$$c^2 d\tau^2 = g_{tt} c^2 dt^2 = c^2 dt^2 \left(1 - \frac{2GM}{rc^2}\right) \tag{232}$$

At $r = r_s$, $g_{tt} \to 0$, so $d\tau/dt \to 0$, and the ratio of emitted to observed frequency (at infinity) is

$$1 + z = \frac{\nu_{\text{emitted}}}{\nu_{\text{observed}}} \to \infty \tag{233}$$

$g_{tt} = 0$ is an *infinite redshift* surface. This, and the event horizon, coincide for a Schwarzschild black hole, but they don't have to (they don't for spinning, 'Kerr' black holes). More generally, an infinite redshift surface occurs where $g_{tt} = 0$.

18.2 Particle geodesics

$L^2 = c^2$, so (225) gives

$$\left(1 - \frac{r_s}{r}\right) c^2 \dot{t}^2 - \dot{r}^2 \left(1 - \frac{r_s}{r}\right)^{-1} = c^2 \tag{234}$$

where now the dot indicates derivative w.r.t. proper time τ. The ELII equation for t gives the same constraint (226), hence

$$\dot{r}^2 = k^2 - c^2 \left(1 - \frac{r_s}{r}\right) \tag{235}$$

For simplicity, choose $k = c$ (particle with zero speed at infinity), for which

$$\begin{aligned} \left(\frac{dr}{d\tau}\right)^2 = \dot{r}^2 &= c^2 \frac{r_s}{r} \\ \Rightarrow c\tau &= \frac{2}{3r_s^{1/2}} \left(r_0^{3/2} - r^{3/2}\right) \end{aligned} \tag{236}$$

where τ is measured from when the particle reaches $r = r_0$. We get the important result that the *proper time* to fall from r_0 to r is **finite**, *even if* $r < r_s$. Nothing singular happens at r_s, as far as an observer travelling through the $r = r_s$ surface is concerned. Note that $r = 0$ is really singular (singular curvature).

In terms of coordinate time t, however,

$$c\frac{dt}{dr} = \frac{c\dot{t}}{\dot{r}} = -\sqrt{\frac{r}{r_s}} \frac{r}{r - r_s} \tag{237}$$

whose integral diverges logarithmically as $r \to r_s$. Since t is the proper time of a stationary observer at infinity, as far as he is concerned, the particle *never crosses* the event horizon.

More on Black Holes at `http://casa.colorado.edu/~ajsh`