Problem Sheet 1

1. The components of T^p_{qr} are :-

$$T^1_{11}, \ T^1_{12}, \ T^1_{21}, \ T^1_{22}, \ T^2_{11}, \ T^2_{12}, \ T^2_{21}, \ T^2_{22}.$$

The components of T^p_{pr} are :

$$T^1_{11} + T^2_{21}, \qquad T^1_{12} + T^2_{22}$$

The components of T^p_{qp} are :

$$T^1_{11} + T^2_{12}, \qquad T^1_{21} + T^2_{22}$$

2. $\delta^p_p = \delta^1_1 + \delta^2_2 + \delta^3_3 + \cdots = 1 + 1 + 1 + \cdots$ (n times) $= n$.

3. $A^{rs} B_{rs} = - A^{sr} B_{rs}$ (since A is antisymmetric)

$= - A^{rs} B_{sr}$ (interchanging r and s, which are dummy suffices)

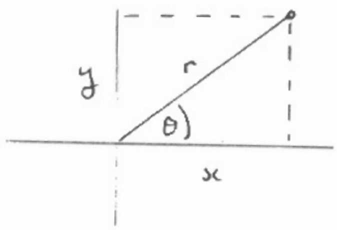
$= - A^{rs} B_{rs}$ (since B symmetric)

$\therefore 2 A^{rs} B_{rs} = 0$

$\therefore A^{rs} B_{rs} = 0.$

4.

$$x = r\cos\theta$$
$$y = r\sin\theta$$



$$dx = \frac{\partial x}{\partial r}\,dr + \frac{\partial x}{\partial \theta}\,d\theta$$

$$dy = \frac{\partial y}{\partial r}\,dr + \frac{\partial y}{\partial \theta}\,d\theta$$

Now $\frac{\partial x}{\partial r} = \cos\theta$, $\frac{\partial x}{\partial \theta} = -r\sin\theta$, $\frac{\partial y}{\partial r} = \sin\theta$, $\frac{\partial y}{\partial \theta} = r\cos\theta$

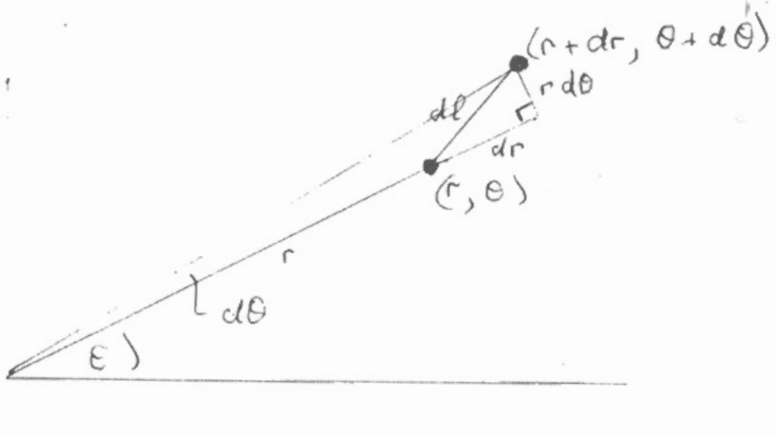
$$dx = \cos\theta\,dr - r\sin\theta\,d\theta$$
$$dy = \sin\theta\,dr + r\cos\theta\,d\theta.$$

$$\therefore\; dx^2 + dy^2 = \cos^2\theta\,dr^2 + r^2\sin^2\theta\,d\theta^2 - \cancel{2r\sin\theta\cos\theta\,dr\,d\theta}$$
$$+ \sin^2\theta\,dr^2 + r^2\cos^2\theta\,d\theta^2 + \cancel{2r\sin\theta\cos\theta\,dr\,d\theta}$$

$$= dr^2\left(\cos^2\theta + \sin^2\theta\right) + r^2 d\theta^2\left(\sin^2\theta + \cos^2\theta\right)$$

ie $dl^2 = dr^2 + r^2 d\theta^2$. where dl = distance between the points (r, θ) and $(r + dr, \theta + d\theta)$.

Geometrically :



Result follows from Pythagoras's theorem applied to the small right angled triangle.

5. $T^{mn} = g^{mp} g^{nq} T_{pq}$

$= g^{mp} g^{nq} T_{qp}$ since T_{pq} symmetric.

$= T^{nm}$ as required.

Also $T^m{}_n = g^{mp} T_{pn} = g^{mp} T_{np} = T_n{}^m$,

again because T is symmetric. In such cases we just write the mixed form of the tensor as T^m_n.

6. $g^{mn} g_{np} = \delta^m{}_p$ is the definition of g^{mn}.

Suppose $g_{np} = 0$ if $p \neq n$. In the summation over n, only the term with $n = p$ contributes, and we get

$g^{mp} g_{pp} = \delta^m{}_p$ (with no summation convention on LHS)

ie $g^{mp} = 0$, $m \neq p$

and $g^{pp} = 1/g^{pp}$

7. In plane polar coordinates $dl^2 = dr^2 + r^2 d\theta^2$

$$= g_{mn} dx^m dx^n$$

where $x^1 = r$, $x^2 = \theta$,

$$g_{11} = 1 \qquad g_{22} = r^2 \qquad g_{12} = g_{21} = 0.$$

Consider now $g^{mn} \frac{\partial}{\partial x^r} g_{mn} = \frac{\partial}{\partial x^r} \ln |g|$.

Here $g = \begin{vmatrix} 1 & 0 \\ 0 & r^2 \end{vmatrix} = r^2$ and $g^{11} = 1 \quad g^{22} = 1/r^2$

$$g^{12} = g^{21} = 0.$$

For $r = 1$, $\text{LHS} = g^{mn} \frac{\partial}{\partial r} g_{mn} = g^{11} \underbrace{\frac{\partial}{\partial r} g_{11}}_{0} + g^{22} \frac{\partial}{\partial r} g_{22} =$

$$= \frac{1}{r^2} \frac{\partial}{\partial r} r^2 = \frac{2r}{r^2} = \frac{2}{r}$$

$$\text{RHS} = \frac{\partial}{\partial r} \ln r^2 = 2 \frac{\partial}{\partial r} \ln r = \frac{2}{r}$$

For $r = 2$ both LHS and RHS $= 0$ since none of the components of g_{mn} and g^{mn} depend on θ.

8. $x^1 = x, \quad x^2 = y$

$x'^1 = r, \quad x'^2 = \theta.$

$$X'^r = \frac{\partial x'^r}{\partial x^s} X^s$$

$$\therefore \quad X'^1 = \frac{\partial x'^1}{\partial x^1} X^1 + \frac{\partial x'^1}{\partial x^2} X^2 = \frac{\partial r}{\partial x} X^1 + \frac{\partial r}{\partial y} X^2$$

But $X^1 = 1$, $X^2 = 0$ so $X'^1 = \frac{\partial r}{\partial x}$. (to be evaluated at the point in question)

Likewise

$$X'^2 = \frac{\partial x'^2}{\partial x^1} X^1 + \frac{\partial x'^2}{\partial x^2} X^2 = \frac{\partial \theta}{\partial x} X^1 + \frac{\partial \theta}{\partial y} X^2 = \frac{\partial \theta}{\partial x}.$$

Now $r = \sqrt{x^2+y^2}$ so $\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2}}$

$\theta = \tan^{-1} y/x$ so $\frac{\partial \theta}{\partial x} = \frac{1}{1+(y/x)^2}\left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2}$

$$\therefore \quad X'^1 = \frac{x}{\sqrt{x^2+y^2}}, \quad X'^2 = -\frac{y}{x^2+y^2}.$$

where the numerical values are ~~to~~ be determined at the point specified in the question (ie inserting the values of x and y).

In the original coordinates,

$$g_{mn} X^m X^n = g_{11}(X^1)^2 + \underset{\overset{\|}{0}}{g_{12}} X^1 X^2 + \underset{\overset{\|}{0}}{g_{21}} X^2 X^1 + g_{22}(X^2)^2 = (X^1)^2 + (X^2)^2 = 1.$$

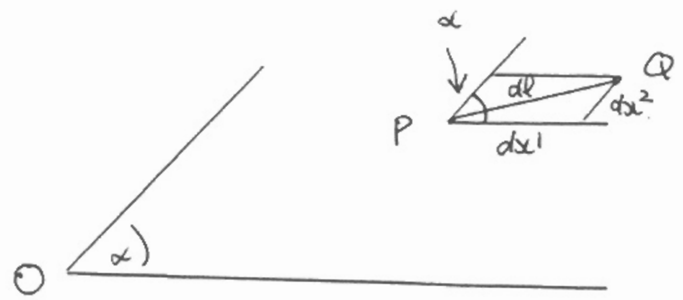
In the dashed coordinates,

$$g'_{mn} X'^m X'^n = g'_{11}(X'^1)^2 + g'_{22}(X'^2)^2 \quad (\text{since } g'_{12} = g'_{21} = 0)$$

$$= (X'^1)^2 + r^2 (X'^2)^2$$

$$= \frac{x^2}{x^2+y^2} + (x^2+y^2)\frac{y^2}{(x^2+y^2)^2} = 1.$$

9.



By cosine rule, $dl^2 = (dx^1)^2 + (dx^2)^2 + 2\, dx^1\, dx^2 \cos\alpha$.

$$= g_{11}(dx^1)^2 + g_{22}(dx^2)^2 + (g_{12} + g_{21})\, dx^1\, dx^2$$

where $g_{11} = 1 \quad g_{22} = 1 \quad g_{12} = g_{21} = \cos\alpha$.

the covariant components are

$$x_\mu = g_{\mu\nu} x^\nu$$

ie $\quad x_1 = g_{11} x^1 + g_{12} x^2 = x^1 + \cos\alpha\, x^2$

$\quad\;\; x_2 = g_{21} x^1 + g_{22} x^2 = \cos\alpha\, x^1 + x^2$.

and we see that these are obtained by drawing perpendicular lines from P to ~~the~~ two axes.