General Relativity

Problem Sheet 3 - solutions

1. For a [2-D] surface,

$$R_{\alpha\mu\sigma\tau} = -R_{\alpha\mu\tau\sigma} \Rightarrow R_{\alpha\mu 11} = R_{\alpha\mu 22} = 0 \qquad \alpha,\mu = 1,2.$$

$$R_{\alpha\mu\sigma\tau} = -R_{\mu\alpha\sigma\tau} \Rightarrow R_{11\sigma\tau} = R_{22\sigma\tau} = 0 \qquad \sigma,\tau = 1,2.$$

Hence non-zero components are

$$R_{1212}, \quad R_{1221}, \quad R_{2112}, \quad R_{2121}$$

But $R_{1212} = -R_{2112} = -R_{1221} = R_{2121}$

$\therefore$ Only the first of these is independent.

2. $ds^2 = c^2 dt^2 - (dx^2 + dy^2 + dz^2)$

From coordinate transformation eqns,

$x' = x - \frac{1}{2} g t^2 \rightarrow dx' = dx - gt\,dt$ or $dx = dx' + gt'dt'$

$dy = dy'$, $dz = dz'$, $dt = dt'$

$$\therefore ds^2 = c^2 dt'^2 - (dx'^2 + g^2 t'^2 dt'^2 + 2gt'dx'dt') - dy'^2 - dz'^2$$

$$= c^2 dt'^2\left(1 - \frac{g^2 t'^2}{c^2}\right) - (dx'^2 + dy'^2 + dz'^2 + 2gt'dx'dt')$$

$$\simeq c^2 dt'^2 - (dx'^2 + dy'^2 + dz'^2 + 2gt'dx'dt') \qquad \text{ignoring quadratic terms}$$

$$= g'_{\mu\nu} dx'^\mu dx'^\nu$$

where $g'_{\mu\nu} = \begin{pmatrix} -1 & & & -\frac{gt'}{c} \\ & -1 & & \\ & & -1 & \\ -\frac{gt'}{c} & & & 1 \end{pmatrix}$

Christoffel symbols:

$$g'_{\mu\nu,\sigma} = 0 \quad \text{except for} \quad g'_{14,4} = g'_{41,4} = -\frac{g}{c^2}$$

Hence, recalling that

$$\Gamma'^{\lambda}_{\mu\nu} = \tfrac{1}{2} g'^{\lambda\kappa} (g'_{\mu\kappa,\nu} + g'_{\nu\kappa,\mu} - g'_{\mu\nu,\kappa})$$

the first task is to calculate $g'^{\lambda\kappa}$, starting from $g'^{\mu\nu} g'_{\nu\sigma} = \delta'^{\mu}_{\sigma}$.

$$\left.\begin{aligned} &\text{Setting } \mu = \sigma = 1, \quad 1 = g'^{11} g'_{11} + g'^{14} g'_{41} = -g'^{11} - \tfrac{gt'}{c} g'^{14} \\ &\mu = 1, \sigma = 4 \quad 0 = g'^{11} g'_{14} + g'^{14} g'_{44} = -\tfrac{gt'}{c} g'^{11} + g'^{14}. \end{aligned}\right\}$$

Solving, $g'^{11} = -\dfrac{1}{1+\frac{g^2t'^2}{c^2}} \simeq -1 \qquad g'^{14} = -\dfrac{gt'}{c} = g'^{41}$

Sim $g'^{22} = g'^{33} = -1, \quad g'^{44} \simeq 1.$

$\therefore$ Components of $g'^{\mu\nu}$ take the same values as $g'_{\mu\nu}$.

Now $g'_{\mu\kappa,\nu} + g'_{\nu\kappa,\mu} - g'_{\mu\nu,\kappa} = 0$ unless two of the suffices μ, κ, ν are equal to 4 and the other is 1.

If $\kappa = \mu = 4, \nu = 1$ we get zero by cancellation

If $\nu = \kappa = 4, \mu = 1$ " " "

If $\mu = \nu = 4, \kappa = 1$ " $g'_{41,4} + g'_{41,4} = -\dfrac{2g}{c^2}$.

$\therefore$ Only non-zero Christoffel symbol are those with $\mu = \nu = 4$

$$\therefore \Gamma'^{\lambda}_{44} = -\frac{g}{c^2} g'^{\lambda 1}$$

$$\therefore \Gamma'^{1}_{44} = \frac{g}{c^2}, \quad \Gamma'^{4}_{44} = \frac{g^2 t'}{c^3}$$

Now $R^{\alpha}_{\mu\sigma\tau} = \Gamma^{\alpha}_{\mu\tau,\sigma} - \Gamma^{\alpha}_{\mu\sigma,\tau} + \Gamma^{\beta}_{\mu\tau}\Gamma^{\alpha}_{\beta\sigma} - \Gamma^{\beta}_{\mu\sigma}\Gamma^{\alpha}_{\beta\tau}$

Recall that $R^{\alpha}_{\mu\sigma\tau}$ is anti-symmetric in σ and τ, so vanishes if $\sigma = \tau$.

But (i) all derivatives of Christoffel symbols vanish except for $\Gamma'^{4}_{44,4}$ for which $\alpha = \mu = \sigma = \tau = 4$

(ii) all products of Christoffel symbols vanish unless $\sigma = \tau = 4$

$\therefore$ $R'^{\alpha}_{\mu\sigma\tau} = 0$ for all combinations of $\alpha, \mu, \sigma, \tau$.

3. From Q 4 in 2nd problem sheet

$$\Gamma^1_{22} = -\sin\theta\cos\theta \qquad \Gamma^2_{12} = \Gamma^2_{21} = \cot\theta.$$

all others zero.

$$g = g_{11} g_{22} = a^4 \sin^2\theta.$$

To verify $\Gamma^\alpha_{\mu\alpha} = \frac{1}{2} \log|g|_{,\mu}$

with $\mu = 1$, $LHS = \cot\theta$, $RHS = \frac{1}{2}\frac{\partial}{\partial\theta}\ln(a^4\sin^2\theta) = \cot\theta.$

$\mu = 2$, $LHS = 0$, $RHS = \frac{1}{2}\frac{\partial}{\partial\phi}(\quad " \quad) = 0.$

The curvature tensor is

$$R^\alpha{}_{\mu\sigma\tau} = \Gamma^\alpha_{\mu\tau,\sigma} - \Gamma^\alpha_{\mu\sigma,\tau} + \Gamma^\beta_{\mu\tau}\Gamma^\alpha_{\beta\sigma} - \Gamma^\beta_{\mu\sigma}\Gamma^\alpha_{\beta\tau}$$

The only non-zero derivatives of Christoffel symbols are

$$\Gamma^1_{22,1} = \frac{\partial}{\partial\theta}(-\sin\theta\cos\theta) = \sin^2\theta - \cos^2\theta.$$

$$\Gamma^2_{12,1} = \Gamma^2_{21,1} = \frac{\partial}{\partial\theta}\cot\theta = -\operatorname{cosec}^2\theta.$$

Since $R^\alpha{}_{\mu\sigma\tau} = 0$ if $\sigma = \tau$, we need only calculate

$$R^1{}_{112} = -R^1{}_{121}, \quad R^1{}_{212} = -R^1{}_{221}, \quad R^2{}_{112} = -R^2{}_{121}, \quad R^2{}_{212} = -R^2{}_{221}.$$

Now $R^1{}_{112} = \Gamma^1_{12,1} - \Gamma^1_{11,2} + \Gamma^\beta_{12}\Gamma^1_{\beta1} - \Gamma^\beta_{11}\Gamma^1_{\beta2} = 0$

$$R^1{}_{212} = \Gamma^1_{22,1} - \Gamma^1_{21,2} + \Gamma^\beta_{22}\Gamma^1_{\beta1} - \Gamma^\beta_{21}\Gamma^1_{\beta2}$$

$$= \sin^2\theta - \cos^2\theta + \sin\theta\cos\theta\cot\theta = \sin^2\theta$$

$$R^2{}_{112} = \Gamma^2_{12,1} - \Gamma^2_{11,2} + \Gamma^\beta_{12}\Gamma^2_{\beta 1} - \Gamma^\beta_{11}\Gamma^2_{\beta 2}$$

$$= -\operatorname{cosec}^2\theta + \cot^2\theta = -1$$

$$R^2{}_{212} = \Gamma^2_{22,1} - \Gamma^2_{21,2} + \Gamma^\beta_{22}\Gamma^2_{\beta 1} - \Gamma^\beta_{21}\Gamma^2_{\beta 2} = 0.$$

$$\therefore \quad R^1{}_{212} = -R^1{}_{221} = \sin^2\theta$$

$$R^2{}_{112} = -R^2{}_{121} = -1$$

all others zero.

Ricci tensor is $\quad R_{\mu\nu} = R^\alpha{}_{\mu\nu\alpha}$.

$$\therefore \quad R_{11} = R^\alpha{}_{11\alpha} = -1$$

$$R_{22} = R^\alpha{}_{22\alpha} = -\sin^2\theta$$

$$R_{12} = R^\alpha{}_{12\alpha} = 0 = R_{21}.$$

and the curvature scalar is

$$R = g^{\mu\nu}R_{\mu\nu} = g^{11}R_{11} + g^{22}R_{22}$$

$$= -\frac{1}{a^2} - \frac{\sin^2\theta}{a^2\sin^2\theta}$$

$$= -\frac{2}{a^2}.$$

4. $$R^{\mu\nu} - \tfrac{1}{2} g^{\mu\nu} R = K T^{\mu\nu}$$

Multiplying by $g_{\mu\nu}$ and noting that $g_{\mu\nu} g^{\mu\nu} = \delta^{\mu}_{\mu} = 4$,

$$R - \tfrac{1}{2}(4) R = K g_{\mu\nu} T^{\mu\nu}$$

$$\therefore \quad R = -K g_{\mu\nu} \left[\left(\rho + \frac{p}{c^2} \right) U^{\mu} U^{\nu} - g^{\mu\nu} p \right]$$

But $$g_{\mu\nu} U^{\mu} U^{\nu} = c^2 g_{\mu\nu} \frac{dx^{\mu}}{ds} \frac{dx^{\nu}}{ds} = c^2$$

$$\Rightarrow \quad R = -K \left[\left(\rho + \frac{p}{c^2} \right) c^2 - 4p \right]$$

$$= -K \left(\rho c^2 - 3p \right)$$

$$= K \left(3p - \rho c^2 \right).$$

5. To show $G^{\mu}{}_{\nu;\mu} = 0$

For $\nu = 1$, this is

$$G^1{}_{1;1} + G^2{}_{1;2} + G^3{}_{1;3} + G^4{}_{1;4} = 0 \quad \text{where}$$

$$G^{\mu}{}_{\nu;\tau} = G^{\mu}{}_{\nu,\tau} + \Gamma^{\mu}_{\sigma\tau} G^{\sigma}{}_{\nu} - \Gamma^{\sigma}_{\nu\tau} G^{\mu}{}_{\sigma} .$$

Since G is a diagonal tensor for the Schwarzschild metric,

$$G^1{}_{1;1} = G^1{}_{1,1} + \Gamma^1_{\sigma 1} G^{\sigma}{}_1 - \Gamma^{\sigma}_{11} G^1{}_{\sigma} = G^1{}_{1,1}$$

$$G^2{}_{1;2} = G^2{}_{1,2} + \Gamma^2_{\sigma 2} G^{\sigma}{}_1 - \Gamma^{\sigma}_{12} G^2{}_{\sigma} = \Gamma^2_{12} (G^1{}_1 - G^2{}_2)$$

$$G^3{}_{1;3} = G^3{}_{1,3} + \Gamma^3_{\sigma 3} G^{\sigma}{}_1 - \Gamma^{\sigma}_{13} G^3{}_{\sigma} = \Gamma^3_{13} (G^1{}_1 - G^3{}_3)$$

equal

$$G^4{}_{1;4} = G^4{}_{1,4} + \Gamma^4_{\sigma 4} G^{\sigma}{}_1 - \Gamma^{\sigma}_{14} G^4{}_{\sigma} = \Gamma^4_{14} (G^1{}_1 - G^4{}_4)$$

$$\therefore G^{\mu}{}_{1;\mu} = \frac{\partial}{\partial r}\left\{\frac{1}{r^2}\left[-1 + e^{-\lambda}(1 + r\nu')\right]\right\} + \frac{2}{r}\left\{\frac{1}{r^2}\left[-1 + e^{-\lambda}(1 + r\nu')\right] - e^{-\lambda}\left(\frac{1}{2}\nu'' + \frac{1}{4}\nu'^2 - \frac{1}{2}\frac{\lambda'}{r} - \frac{1}{4}\lambda'\nu' + \frac{1}{2}\frac{\nu'}{r}\right)\right\} + \frac{1}{2}\nu' e^{-\lambda}\frac{\lambda' + \nu'}{r}$$

the terms arising from the $\frac{\partial}{\partial r}\left(\frac{1}{r^2}\right)$ factor exactly cancel the first of the terms within the second curly bracket on the RHS

$$\therefore G^{\mu}{}_{1;\mu} = \frac{1}{r^2}\left[-\lambda' e^{-\lambda}(1 + r\nu') + e^{-\lambda}(\nu' + r\nu'')\right] - \frac{2}{r} e^{-\lambda}\left(\frac{1}{2}\nu'' + \frac{1}{4}\nu'^2 - \frac{1}{2}\frac{\lambda'}{r} - \frac{1}{4}\lambda'\nu' + \frac{1}{2}\frac{\nu'}{r}\right) + \frac{1}{2}\nu' e^{-\lambda}\frac{\lambda' + \nu'}{r}$$

$$= \frac{e^{-\lambda}}{r^2}\left[-\lambda'(1 + r\nu') + \nu' + r\nu'' - \frac{1}{2}r\nu'^2 + \lambda' + \frac{1}{2}r\lambda'\nu' - \nu' + \frac{1}{2}\nu' r\lambda' + \frac{1}{2}r\nu'^2\right]$$

$= 0$ since all terms cancel.

6. For light signal from A to B, the coordinates of the emission and receipt of the signal are

$$(x^1, x^2, x^3, x^4) \quad \text{and} \quad (x^1+dx^1, x^2+dx^2, x^3+dx^3, x^4+dx_1^4) \quad \text{say.}$$

ie the displacement $dx^\mu = (dx^1, dx^2, dx^3, dx_1^4)$.

Since $ds = 0$, $g_{\mu\nu}\, dx^\mu dx^\nu = 0$

ie
$$g_{ij}\, dx^i dx^j + 2 g_{4j}\, dx_1^4 dx^j + g_{44}(dx_1^4)^2 = 0. \qquad (1)$$

Similarly for the light signal from B to A, the corresponding displacement is

$$dx^\mu = (-dx^1, -dx^2, -dx^3, dx_2^4)$$

So $ds = 0$ gives

$$g_{ij}\, dx^i dx^j - 2 g_{4j}\, dx_2^4 dx^j + g_{44}(dx_2^4)^2 = 0 \qquad (2).$$

Solving (1) and (2) for dx_1^4 and dx_2^4 respectively,

$$dx_1^4 = \frac{1}{g_{44}}\left[-g_{4j}\, dx^j \pm \sqrt{g_{4j} g_{4i}\, dx^i dx^j - g_{44} g_{ij}\, dx^i dx^j}\right]$$

$$dx_2^4 = \frac{1}{g_{44}}\left[g_{4j}\, dx^j \pm \sqrt{g_{4j} g_{4i}\, dx^i dx^j - g_{44} g_{ij}\, dx^i dx^j}\right]$$

∴ Taking +ve sign in each case, since g_{44} must be positive,

$$(dx_1^4 + dx_2^4) = \frac{2}{g_{44}}\sqrt{(g_{4j} g_{4i} - g_{44} g_{ij})\, dx^i dx^j}$$

$$\therefore\ d\tau = \frac{1}{c}(g_{44})^{\frac{1}{2}}(dx_1^4 + dx_2^4) = \frac{2}{c}\sqrt{\left(\frac{g_{4j} g_{4i}}{g_{44}} - g_{ij}\right) dx^i dx^j} = \frac{2\,dl}{c}$$

where $dl = \sqrt{\left(\dfrac{g_{4j} g_{4i}}{g_{44}} - g_{ij}\right) dx^i dx^j}$ is the distance between A and B.

7. Define new polar coordinates (r, θ, ϕ) given by

$$x = r \sin\theta \cos\phi$$
$$y = r \sin\theta \sin\phi$$
$$z = r \cos\theta .$$

Then $x^2+y^2+z^2 = r^2$ and $dx^2+dy^2+dz^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2$.

$$\therefore \quad dl = \frac{(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2)^{1/2}}{1 + r^2/4a^2}$$

$\therefore$ Radius of sphere "$r = \text{const}$" is

$$\rho = \int_0^r \frac{dr}{1 + r^2/4a^2} \simeq \int_0^r dr\left(1 - \frac{r^2}{4a^2}\right) = r\left(1 - \frac{r^2}{12a^2}\right)$$

In θ and ϕ directions, distances are $\dfrac{r\,d\theta}{1+r^2/4a^2}$ and $\dfrac{r\sin\theta\, d\phi}{1+r^2/4a^2}$

Hence area of sphere "$r = \text{const}$" is

$$S = \int_0^{2\pi}\int_0^{\pi} \frac{r^2 \sin\theta \, d\theta \, d\phi}{\left(1 + \frac{r^2}{4a^2}\right)^2} = \frac{4\pi r^2}{(1 + r^2/4a^2)^2}$$

$$\simeq 4\pi r^2\left(1 - \frac{r^2}{2a^2}\right) \simeq \frac{4\pi\rho^2}{\left(1 - \frac{r^2}{12a^2}\right)^2}\left(1 - \rho^2/2a^2\right)$$

$$\simeq 4\pi\rho^2\left(1 - \tfrac{1}{2}\rho^2/a^2 + \tfrac{1}{6}\rho^2/a^2\right)$$

$$= 4\pi\rho^2\left(1 - \tfrac{1}{3}\rho^2/a^2\right) \qquad \therefore \quad A = \frac{1}{3a^2}.$$

Also, volume of sphere "$r = \text{const}$" is

$$V = \int_0^r\int_0^{2\pi}\int_0^{\pi} \frac{r^2\sin\theta \, d\theta \, d\phi \, dr}{\left(1 + \frac{r^2}{4a^2}\right)^3} \simeq 4\pi\int_0^r r^2 dr\left(1 - \tfrac{3}{4} r^2/a^2\right)$$

$$= \tfrac{4}{3}\pi r^3\left(1 - \tfrac{9}{20} r^2/a^2\right)$$

Substituting $r = \rho/(1 - r^2/12a^2)$ as before, this comes to $V = \frac{4}{3}\pi\rho^3\left(1 - \frac{1}{5}\rho^2/a^2\right)$,

ie $B = \dfrac{1}{5a^2}$.

8. The orbit equation is $\dfrac{d^2u}{d\phi^2} + u = \dfrac{mc^2}{h^2} + 3mu^2$.

For circular orbits $u = \frac{1}{r}$ is constant, hence $3mu^2 - u + \dfrac{mc^2}{h^2} = 0$.

$$\therefore u = \frac{1 \pm \sqrt{1 - \frac{12m^2c^2}{h^2}}}{6m}$$

For a real solution we require $\underline{\dfrac{\sqrt{12}\, mc}{h} < 1}$

and we also note that two radii are allowed for each value of angular momentum. The inner orbital corresponds to the upper choice of sign, and u must lie between $\frac{1}{6m}$ and $\frac{1}{3m}$, ie $3m \leq r \leq 6m$. The outer orbital corresponds to the lower sign and we have $0 \leq u \leq \frac{1}{6m}$ ie $6m \leq r \leq \infty$.

<u>Calculation of period of orbit.</u>

$$h = cr^2 \frac{d\phi}{ds} = cr^2 \frac{d\phi}{dt}\frac{dt}{ds} = \frac{c}{u^2}\frac{d\phi}{dt}\frac{k}{1-2mu}$$

$$\therefore \frac{d\phi}{dt} = \frac{hu^2(1-2mu)}{ck}$$ Now develop expressions for h and k.

From eqn above $h^2 = \dfrac{mc^2}{u - 3mu^2} \Rightarrow h = \dfrac{mc}{\sqrt{mu - 3m^2u^2}}$

Also from

$$\frac{1}{r^4}\left(\frac{dr}{d\phi}\right)^2 = -\left(1 - \frac{2m}{r}\right)\frac{c^2}{h^2} - \left(1 - \frac{2m}{r}\right)\frac{1}{r^2} + \frac{k^2c^4}{h^2},$$

We have, since LHS $= 0$,
$$k^2 = \frac{h^2}{c^4}(1-2mu)\left(\frac{c^2}{h^2} + u^2\right)$$
$$= \frac{1}{c^2}(1-2mu)\left(1 + \frac{u^2h^2}{c^2}\right)$$
$$= \frac{1}{c^2}(1-2mu)\left(1 + \frac{u^2m^2}{mu - 3m^2u^2}\right)$$

$$\therefore k^2c^2 = \frac{(1-2mu)^2}{(1-3mu)}$$

$$\therefore \frac{d\phi}{dt} = u^2(1-2mu)\frac{mc}{\sqrt{mu}\sqrt{1-3mu}}\frac{\sqrt{1-3mu}}{1-2mu}$$

$$= u^{3/2} m^{\frac{1}{2}} c$$

$\therefore$ Period of orbit is $\dfrac{2\pi}{u^{3/2} m^{\frac{1}{2}} c} = \dfrac{2\pi r^{3/2}}{m^{\frac{1}{2}} c} = \dfrac{2\pi r^{3/2}}{\sqrt{GM}}$ according to coordinate clocks.

Since $d\tau = \sqrt{1-\frac{2m}{r}}\,dt$, period acc. to standard clocks is $\dfrac{2\pi r^{3/2}}{\sqrt{GM}}\sqrt{1-\frac{2m}{r}}$

Newtonian result:

For circular orbit $\dfrac{mv^2}{r} = \dfrac{GmM}{r^2}$

$$\therefore r^2\left(\frac{d\phi}{dt}\right)^2 = \frac{GM}{r}$$

$$\therefore \frac{d\phi}{dt} = \frac{\sqrt{GM}}{r^{3/2}}$$

$\therefore$ Period of orbit is $\dfrac{2\pi r^{3/2}}{\sqrt{GM}}$.

So the coordinate time for 1 revolution around a circular orbit is the same in Schwarzschild space-time as in flat space-time

Velocity of particle

$$v = \frac{r\frac{d\phi}{dt}}{\sqrt{1-\frac{2m}{r}}} = \frac{u^{\frac{1}{2}} m^{\frac{1}{2}} c}{\sqrt{1-2mu}}$$

$$\therefore \frac{v}{c} = \sqrt{\frac{mu}{1-2mu}}$$

Now RHS < 1 provided $mu < \frac{1}{3}$, so we see the reason for the absolute restriction to $3mu < 1$ (ie $r > 3m$)

Note: corresponding classical result is $v = r\frac{d\phi}{dt} = \sqrt{GMu} = \sqrt{mu}\,c$

i.e. $v/c = \sqrt{mu}$

9. (a) $$ds^2 = c^2 d\bar{t}^2 - (d\bar{r}^2 + \bar{r}^2 d\bar{\theta}^2)$$

$$= c^2 dt^2 - (dr^2 + r^2 (d\theta + \omega\, dt)^2)$$

$$= c^2 dt^2 - (dr^2 + r^2 d\theta^2 + \omega^2 r^2 dt^2 + 2\omega r^2 d\theta\, dt)$$

$$= c^2 dt^2 (1 - \omega^2 r^2/c^2) - (dr^2 + r^2 d\theta^2 + 2\omega r^2 d\theta\, dt).$$

(b) A reference point has a fixed value of r and θ.

$\therefore \bar{r} = r,\ \bar{\theta} = \theta + \omega t = \theta + \omega \bar{t}$

show that, in the reference frame $\bar{S}$, $\bar{r}$ is constant and $\bar{\theta}$ increases linearly with $\bar{t}$. So the reference point (r, θ) rotates with angular velocity ω about the axis of rotation. The same is true for all such reference points, so the frame of reference S can be identified with the rotating disc.

(c) $$d\tau = \frac{1}{c} (g_{44})^{1/2} dx^4$$

Here $dx^4 = c\,dt$, $g_{44} = 1 - \omega^2 r^2/c^2$

So $$d\tau = (1 - \omega^2 r^2/c^2)^{1/2} dt$$

Likewise, $$dl^2 = \left(- g_{jk} + \frac{g_{4j} g_{4k}}{g_{44}}\right) dx^j dx^k \qquad j, k = 1, 2$$

Now $$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -r^2 & -\omega r^2/c \\ 0 & -\frac{\omega r^2}{c} & 1 - \frac{\omega^2 r^2}{c^2} \end{pmatrix} \qquad \mu, \nu = 1, 2, 4$$

(3rd spatial dimension has been suppressed)

$$\therefore \; dl^2 = -g_{jk}\, dx^j dx^k + \frac{g_{4j} g_{4k}\, dx^j dx^k}{g_{44}}$$

$$= dr^2 + r^2 d\theta^2 + \frac{(g_{42})^2}{g_{44}} d\theta^2$$

$$= dr^2 + d\theta^2 \left(r^2 + \frac{\omega^2 r^4/c^2}{1-\omega^2 r^2/c^2} \right)$$

$$= dr^2 + \frac{r^2 d\theta^2}{1-\omega^2 r^2/c^2}$$

(d) $d\tau = (1-\omega^2 r^2/c^2)^{1/2} dt = \sqrt{1-v^2/c^2}\, dt$ since $v = \omega r$.

$d\tau$ = time interval between 2 events occurring at the same place on the disc

$dt = d\bar{t}$ = time interval between them as measured in non-rotating frame.

$d\tau < dt$ means moving clock goes slow by usual SR factor.

$$dl_r = dr, \quad dl_\theta = \frac{r\, d\theta}{\sqrt{1-\omega^2 r^2/c^2}}$$

No contraction in radial direction

$$r\, d\theta = \sqrt{1-v^2/c^2}\, dl_\theta$$

Length between (r,θ) and $(r, \theta + d\theta)$ according to observers in inertial frame is $dl_\theta \times$ contraction factor $\sqrt{1-v^2/c^2}$

(e) Circumference $= \int dl_\theta = \int_0^{2\pi} \frac{r\, d\theta}{\sqrt{1-\omega^2 r^2/c^2}} = \frac{2\pi r}{\sqrt{1-\omega^2 r^2/c^2}}$