[redacted]

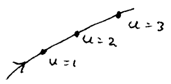
Some basic concepts:

Event. : an extensionless occurrence, specified by coordinates $x^\mu \equiv (x^1, x^2, x^3, x^4)$

(Comment: we don't specify here how these are to be measured; similar to generalised coordinates q_i in classical mechanics, and contrary to the dogma of operationalism. No restriction to 3 space and 1 time coord — could have 4 time coords.)

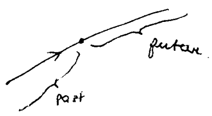
Space-time : Totality of all events (real or imagined)

Particle: a moving point, endowed with mass. Traces out a path in space-time $x^\mu = x^\mu(u)$ called a world-line, the parameter u specifying a point on the world line.



Given a fixed event on a world line, we assume we can divide all other events on the world line into

past : all events happening earlier
future : " " " later

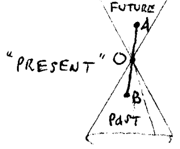


Null Cone

We retain the belief that a particle can't be in two places at the same time or (more precisely:) some pairs of events can't be linked by the world-line of any particle.

For any event O, all other events can be divided into 2 classes:
events which can be linked to O via a particles' world line
" " can't " " " " " "

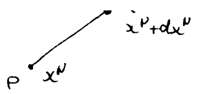
These two classes occupy distinct "volumes" in space-time, separated by a "surface" called the null cone. Schematically:



A happens after O on any world-line through O
B " before O " " "

Definition

Given two neighbouring events:



Displacement dx^μ is "timelike" if P and Q can be linked by a particle's world line
"null" " " " " photon's " "
"space-like" otherwise.

Note: null displacements from O in previous diagram lie on surface of null cone, hence name "null cone".

Postulate I(a)

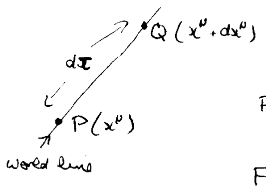
Space-time is Riemannian, ie there exists a metric tensor $g_{\mu\nu}$ $(= g_{\nu\mu})$ whose components are functions of the coordinates x^μ.

Defn The "interval" ds between x^μ and $x^\mu + dx^\mu$ is given by

$$\underline{ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu}$$

Postulate I(b)

For a time-like displacement $g_{\mu\nu} dx^\mu dx^\nu > 0$, and $\frac{ds}{c} = \sqrt{g_{\mu\nu} dx^\mu dx^\nu}/c$ is the elapsed time between x^μ and $x^\mu + dx^\mu$ along the world line joining them, ie in the frame in which the two events occur at the same place.



[Comment: $\frac{ds}{c}$ is measured by a standard clock. It is the "proper-time" between P and Q, and denoted by $d\tau$]

For a null displacement $g_{\mu\nu} dx^\mu dx^\nu = 0$ ie $ds = 0$

For a space-like displacement $g_{\mu\nu} dx^\mu dx^\nu < 0$ ie ds is imaginary.

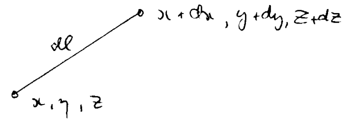
Note

1. $ds = 0$ divides space-like from time-like displacements. Hence $ds = 0$ for any two neighbouring points on the null cone.

2. Exp. verification of I(a) and I(b): Take fixed event P, and arrange for 10 observers to move along 10 different world lines starting from P. Each move from P to a neighbouring point and measures ds/c. Since $g_{\mu\nu}$ has 10 independent components, $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$ gives us 10 equns. and 10 unknowns. Solve for $g_{\mu\nu}$ (at P). Now get 11 th observer to move from P to an arbitrary neighbouring point Q. Measure ds, and compare with calculated value $\sqrt{g_{\mu\nu} dx^\mu dx^\nu}$.

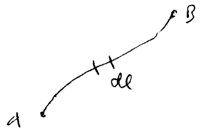
Geodesics

Euclidean 3-space $\quad dl^2 = dx^2 + dy^2 + dz^2$



Given two fixed points A and B,

$\delta \int_A^B dl = 0$ is the equation of a straight line



ie gives shortest distance between A + B.

Defn. By analogy, a geodesic is a curve in spacetime whose "length" has a stationary value wrt small variations, end points being fixed.

$$\text{ie} \quad \underline{\delta \int_A^B ds = 0.}$$

Now any curve between A and B may be ~~defined~~ described parametrically by $x^\mu = x^\mu(u)$ where u is a parameter

$$\therefore \; 0 = \delta \int_A^B ds = \delta \int_A^B \sqrt{g_{\mu\nu} \frac{dx^\mu}{du} \frac{dx^\nu}{du}}\, du = \delta \int_A^B L\, du$$

where $\quad L = \sqrt{g_{\mu\nu} \frac{dx^\mu}{du} \frac{dx^\nu}{du}} = L\left(x^\kappa, \frac{dx^\kappa}{du}\right).$

Now have analysis analogous to deriving Lagrange's equations from Hamilton's principle:

$$0 = \int_A^B \delta L\, du = \int_A^B \left(\frac{\partial L}{\partial x^\kappa} \delta x^\kappa + \frac{\partial L}{\partial \left(\frac{dx^\kappa}{du}\right)} \delta\left(\frac{dx^\kappa}{du}\right) \right) du$$

$$= \int_A^B \left(\frac{\partial L}{\partial x^\kappa} \delta x^\kappa + \frac{\partial L}{\partial \left(\frac{dx^\kappa}{du}\right)} \frac{d}{du} \delta x^\kappa \right) du$$

$$= \underbrace{\left. \frac{\partial L}{\partial \left(\frac{dx^\kappa}{du}\right)} \delta x^\kappa \right|_A^B}_{=0} + \int_A^B \left[\frac{\partial L}{\partial x^\kappa} - \frac{d}{du} \frac{\partial L}{\partial \left(\frac{dx^\kappa}{du}\right)} \right] \delta x^\kappa\, du.$$

Since true for arbitrary variations between A and B,

$$\frac{\partial L}{\partial x^\kappa} - \frac{d}{du} \frac{\partial L}{\partial \left(\frac{dx^\kappa}{du}\right)} = 0. \qquad \text{(analogue of Lagrange's eqns)}$$

Now $$\frac{\partial L}{\partial x^\kappa} = \frac{1}{2L}\frac{dx^\mu}{du}\frac{dx^\nu}{du}\frac{\partial g_{\mu\nu}}{\partial x^\kappa}$$

$$= \frac{1}{2L}\frac{dx^\mu}{du}\frac{dx^\nu}{du} g_{\mu\nu,\kappa}$$

where $(\ )_{,\kappa}$ means $\frac{\partial}{\partial x^\kappa}(\ )$.

and $$\frac{\partial L}{\partial\left(\frac{dx^\kappa}{du}\right)} = \frac{1}{2L}\frac{\partial}{\partial\left(\frac{dx^\kappa}{du}\right)} g_{\mu\nu}\frac{dx^\mu}{du}\frac{dx^\nu}{du}$$

$$= \frac{1}{2L}\left(g_{\kappa\nu}\frac{dx^\nu}{du} + g_{\mu\kappa}\frac{dx^\mu}{du}\right)$$

$$= \frac{1}{L} g_{\mu\kappa}\frac{dx^\mu}{du}$$

$$\therefore \frac{d}{du}\left[\frac{1}{L} g_{\mu\kappa}\frac{dx^\mu}{du}\right] - \frac{1}{2L}\frac{dx^\mu}{du}\frac{dx^\nu}{du} g_{\mu\nu,\kappa} = 0$$

Now let $u = s$, provided $ds \neq 0$ ie we are not dealing with a null geodesic.

Then $L = 1$ and $$\frac{d}{ds}\left(g_{\mu\kappa}\frac{dx^\mu}{ds}\right) - \frac{1}{2}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} g_{\mu\nu,\kappa} = 0.$$

ie $$g_{\mu\kappa}\frac{d^2x^\mu}{ds^2} + g_{\mu\kappa,\nu}\frac{dx^\nu}{ds}\frac{dx^\mu}{ds} - \frac{1}{2} g_{\mu\nu,\kappa}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 0$$

But $$g_{\mu\kappa,\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = g_{\nu\kappa,\mu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}$$

$$\therefore g_{\mu\kappa}\frac{d^2x^\mu}{ds^2} + \frac{1}{2}\left(g_{\mu\kappa,\nu} + g_{\nu\kappa,\mu} - g_{\mu\nu,\kappa}\right)\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 0.$$

$\times\, g^{\lambda\kappa}$ and use $g^{\lambda\kappa} g_{\mu\kappa} = \delta^\lambda_\mu \Rightarrow$

$$\underline{\frac{d^2x^\lambda}{ds^2} + \Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 0}$$

where $\underline{\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\kappa}\left(g_{\mu\kappa,\nu} + g_{\nu\kappa,\mu} - g_{\mu\nu,\kappa}\right)}$. are <u>Christoffel symbols</u>.

<u>Special case – null geodesics</u>   A null geodesic is a path for which the interval ds between neighbouring points is zero.

Here $ds = 0$ so can't use s as parameter in $x^\mu = x^\mu(s)$

Procedure: $g_{\mu\nu}\, dx^\mu dx^\nu = 0$, since $ds = 0$ and $x^\mu = x^\mu(u)$ $u \neq s$.

$$\delta \int_A^B g_{\mu\nu} \frac{dx^\mu}{du}\frac{dx^\nu}{du}\, du = 0$$

ie $\delta \int_A^B L\, du = 0$ where $L = g_{\mu\nu} \frac{dx^\mu}{du}\frac{dx^\nu}{du}$

Analysis same as before, except no $\frac{1}{2L}$ factor in $\frac{\partial L}{\partial x^\kappa}$ and $\frac{\partial L}{\partial\left(\frac{dx^\kappa}{du}\right)}$

Gives eventually $\frac{d^2x^\lambda}{du^2} + \Gamma^\lambda_{\mu\nu} \frac{dx^\mu}{du}\frac{dx^\nu}{du} = 0$.

<u>Properties of Christoffel symbols</u>

$[\mu\nu, \kappa] = [\nu\mu, \kappa]$ by inspection.

$[\mu\nu, \kappa] + [\kappa\nu, \mu] = \frac{1}{2}(g_{\mu\kappa,\nu} + g_{\nu\kappa,\mu} - g_{\mu\nu,\kappa}) + \frac{1}{2}(g_{\kappa\mu,\nu} + g_{\nu\mu,\kappa} - g_{\kappa\nu,\mu})$

$= g_{\mu\kappa,\nu}$

(1) $\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu}$, by inspection.

$g_{\lambda\kappa}\left\{ {\lambda \atop \mu\,\nu} \right\} = g_{\lambda\kappa}\, g^{\lambda\sigma} [\mu\nu, \sigma] = \delta^\sigma_\kappa [\mu\nu, \sigma] = [\mu\nu, \kappa]$

$\therefore$ From $g_{\mu\kappa,\nu} = [\mu\nu, \kappa] + [\kappa\nu, \mu]$, we deduce

(2) $\therefore$ $g_{\mu\kappa,\nu} = g_{\lambda\kappa} \Gamma^\lambda_{\mu\nu} + g_{\lambda\mu} \Gamma^\lambda_{\kappa\nu}$

But bear in mind: Christoffel symbols are <u>not</u> tensors (see later) but simply functions of the metric tensor and its derivatives.

Postulate II   Free particles (ones on which no real forces act) follow time-like geodesics through space-time. Photons follow null geodesics.

Hence

Particles

Equn of motion is

$$\frac{d^2x^\lambda}{ds^2} + \Gamma^\lambda_{\mu\nu} \frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 0$$

with particular integral

$$g_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 1$$

Solution $x^\mu(s)$ requires initial conditions on x^μ, $\frac{dx^\mu}{ds}$

Photons

Equn of motion is

$$\frac{d^2x^\lambda}{du^2} + \Gamma^\lambda_{\mu\nu} \frac{dx^\mu}{du}\frac{dx^\nu}{du} = 0$$

with particular integral

$$g_{\mu\nu}\frac{dx^\mu}{du}\frac{dx^\nu}{du} = 0$$

Solution $x^\mu(u)$ requires initial conditions on x^μ, $\frac{dx^\mu}{du}$

Notes:

① Motivation for making this postulate: In Newtonian mechanics, free particles move in straight lines. In space-time with general metric, the analogue of a straight line is a geodesic.

② Postulate that photons follow null geodesics means that the null cone introduced earlier is a surface determined by the world lines of photons converging on O and diverging from O.

Special Relativity

We now confirm the validity of the above postulates for those frames of ref. (the inertial frames) in which Sp. Rel. holds. (locally).

Cartesian spatial coords. x, y, z.

$\therefore$ Define $x^1 = x \quad x^2 = y \quad x^3 = z \quad x^4 = ct$.

Postulate I (a). – existence of metric tensor:

$c^2dt^2 - (dx^2 + dy^2 + dz^2)$ is invariant under transformation between inertial frames

$\therefore$ Write $ds^2 = c^2dt^2 - (dx^2 + dy^2 + dz^2) = (dx^4)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 = g_{\mu\nu}\, dx^\mu dx^\nu$

$$\text{where} \quad g_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix}$$

Note: we assume that $g_{\mu\nu}(x^\kappa) = (-1, -1, -1, +1)$ for all x^κ, even though we know that this will be so only over a limited region of spacetime.

Postulate I(b) – meaning of ds.

$$\frac{ds}{c} = \sqrt{dt^2 - \frac{dx^2 + dy^2 + dz^2}{c^2}} = dt \text{ in that frame of reference for which } dx = dy = dz$$

ie in frame in which the two events occur at the same place.

Postulate II – geodesics.

Since $g_{\mu\nu}$ = constant, all $\Gamma^\lambda_{\mu\nu}$ are zero.

Free particles: $\dfrac{d^2x^\lambda}{ds^2} = 0 \Rightarrow x^\lambda = \alpha^\lambda s + \beta^\lambda$ where $\alpha^\lambda, \beta^\lambda$ are constants.

Velocity of particle in x-direction is

$$v_x = \frac{dx}{dt} = c\frac{dx^1}{dx^4} = c\,\frac{dx^1}{ds}\Big/\frac{dx^4}{ds} = c\frac{\alpha^1}{\alpha^4} = \text{const.}$$

Similarly v_y = const, v_z = const. $\therefore$ particles move with const. velocity.

Particular integral is $g_{\mu\nu}\dfrac{dx^\mu}{ds}\dfrac{dx^\nu}{ds} = 1$

$$\text{ie } \underline{\frac{1}{c}\frac{ds}{dt} = \sqrt{1 - v^2/c^2}} \quad \text{where } v^2 = v_x^2 + v_y^2 + v_z^2$$

This is the time-dilation formula.

For photons, $\frac{d^2x^\lambda}{du^2} = 0 \Rightarrow$ photons have constant velocity

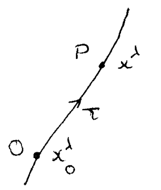
Particular integral $g_{\mu\nu}\frac{dx^\mu}{du}\frac{dx^\nu}{du} = 0 \Rightarrow dt^2 = \frac{dx^2+dy^2+dz^2}{c^2}$

ie $v = c$.

Hence postulates correct. But note severe restriction to inertial frames. In a non-inertial frame, even with cartesian coords, we're not going to find $g_{\mu\nu} = \begin{pmatrix}-1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1\end{pmatrix}$ and hence free particles are not going to move with const. velocity.

Riemannian coordinates (sometimes called geodesic coordinates)

Aim: starting from an arbitrary coordinate system, to find a coordinate system in which S.R holds locally.



O is a fixed point, with coords x^λ_0.

OP is an arbitrary time-like geodesic whose equation is

$$\frac{d^2x^\lambda}{ds^2} + \Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 0 \quad \text{or} \quad \frac{d^2x^\lambda}{d\tau^2} + \Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0 \quad \text{where} \quad d\tau = \frac{ds}{c} \text{ is proper time}$$

Suppose we measure s and therefore τ from O. Then $x^\lambda = x^\lambda(\tau)$ along geodesic.

Expanding about O,

$$x^\lambda = x^\lambda_0 + \left(\frac{dx^\lambda}{d\tau}\right)_0\tau + \frac{1}{2}\left(\frac{d^2x^\lambda}{d\tau^2}\right)_0\tau^2 + \frac{1}{6}\left(\frac{d^3x^\lambda}{d\tau^3}\right)_0\tau^3 + \dots$$

Defining $U^\lambda = \frac{dx^\lambda}{d\tau}$, the velocity vector

$$x^\lambda = x^\lambda_0 + U^\lambda_0\tau + \frac{1}{2}\left(\frac{dU^\lambda}{d\tau}\right)_0\tau^2 + \frac{1}{6}\left(\frac{d^2U^\lambda}{d\tau^2}\right)_0\tau^3 + \dots$$

But $\frac{dU^\lambda}{d\tau} = -\Gamma^\lambda_{\mu\nu}U^\mu U^\nu$.

$$\frac{d^2U^\lambda}{d\tau^2} = -\Gamma^\lambda_{\mu\nu,\rho}U^\mu U^\nu U^\rho - \Gamma^\lambda_{\mu\nu}U^\mu\left[-\Gamma^\nu_{\sigma\rho}U^\sigma U^\rho\right]$$
$$-\Gamma^\lambda_{\mu\nu}U^\nu\left[-\Gamma^\mu_{\sigma\rho}U^\sigma U^\rho\right]$$

$-\Gamma^\lambda \quad U^\mu U^\nu U^\rho \quad \dots$