5.5 Energy-momentum tensor

\[ F_i = \sigma_{ij} A_j \]

\( \sigma_{ij} \) components of stress tensor

\[ F_i \text{ force} \]
\[ A \text{ area of cross-section} \]
\[ n \text{ normal to cross-section} \]
\[ A_j = A (n \cdot e_j) \]

provides relation between the forces and the cross-sections these are exerted on

for fluid in thermodynamic equilibrium:

\[ \sigma_{ij} = P \delta_{ij} \quad P \text{ pressure} \]

(no shear stresses)

complement \( \sigma_{ij} \) to \( T^{\mu\nu} \)

\[ T^{00} \text{ energy density} \]
\[ T^{0i} = T^{i0} \text{ momentum density} \]
\[ T^{ij} \text{ stress} \]
\[ c \text{ mass density} \]

in fluid rest frame:

\[ \tilde{T}^{00} = \rho c^2 \quad \tilde{T}^{0i} = \tilde{T}^{i0} = 0 \quad \tilde{T}^{ij} = P \delta^{ij} \]

\[ T^{\mu\nu} = \left( \rho + \frac{P}{c^2} \right) u^\mu u^\nu - P g^{\mu\nu} \text{ energy-momentum tensor} \]
\[ T^{\mu\nu} = \left( \rho + \frac{P}{c^2} \right) u^\mu u^\nu - P g^{\mu\nu} \]

\[ T^\nu_\mu = T^{\mu\nu} \quad T^{\mu\nu}_{\ ;\nu} = 0 \]

non-relativistic limit: \( v \ll c \quad P \ll \rho c^2 \)

\[ T^{00} \approx \rho c^2 \]
\[ T^{0i} \approx \rho c v^i \]
\[ T^{ij} \approx \rho v^i v^j + P \delta^{ij} \]

\[ T^{\mu\nu}_{\ ;\nu} = 0 \iff \frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{v}) \quad \int \frac{\partial \rho}{\partial t} \, dV = -\int \rho \mathbf{v} \cdot d\mathbf{S} \quad \text{(continuity equation)} \]
\[ \rho \frac{\mathrm{d} \mathbf{v}}{\mathrm{d} t} = -\nabla P \quad \text{(\leftrightarrow Newton’s law)} \]
6 General Relativity

6.1 Principles

experiments cannot distinguish between:
- virtual forces present in non-inertial frames
- true forces

non-inertial frames can be described by
space-time metric \( ds^2 = g_{\mu\nu} dq^\mu dq^\nu \)

true forces can be described by
space-time metric \( ds^2 = g_{\mu\nu} dq^\mu dq^\nu \)

gravitation becomes property of space-time
with particles moving on geodesics
local free-falling frame is inertial frame,
where
\[ ds^2 = c^2 dt^2 - dr^2 \]

only remaining issue: relation between \( ds^2 \) and Newton’s law
→ Einstein’s field equations
General Relativity summarized in 6 points

The laws of physics are the same for all observers, irrespective of their motion

The laws of physics take the same form in all coordinate systems

We live in a 4-dimensional curved metric space-time

The curvature follows the energy-momentum tensor as described by Einstein’s field equations

The laws of Special Relativity apply locally for all inertial observers

Particles move along geodesics
6.2 Einstein’s field equations

independence on choice of coordinates
- formulate theory by means of tensor fields

matter is completely described by 2nd-rank tensor
\{ energy-momentum tensor \}

\[ T^\nu_\mu = T^\mu_\nu \quad T^\mu_\nu ;\nu = 0 \]

description of curvature by 2nd-rank tensor
\{ Einstein tensor \}

\[ G^\nu_\mu = G^\mu_\nu \quad G^\mu_\nu ;\nu = 0 \]

\[ G^\mu_\nu = \kappa T^\mu_\nu \]

if non-relativistic limit reproduces Newton’s law,
- this is not necessarily the only possible theory,
  but the most simple one that conforms to the principles
non-relativistic limit ($v \ll c$, $P \ll \rho c^2$):

$$T^{00} \simeq \rho c^2$$ dominating

$$g_{00} \simeq 1 + \frac{2 \Phi}{c^2}$$

Einstein’s field equations:

$$R_{\mu \nu} = \kappa \left( T_{\mu \nu} - \frac{1}{2} \delta_{\mu \nu} T \right) = 0$$

$$T = T_{\mu}^{\mu} \simeq T_{0}^{0} \simeq T^{00}$$

$$R_{00} = R_{0}^{0} \simeq \frac{\kappa}{2} \rho c^2$$
\[ R_{\mu \nu} = \Gamma^\rho_{\mu \nu, \rho} - \Gamma^\rho_{\mu \rho, \nu} + \Gamma^\rho_{\mu \nu} \Gamma^\sigma_{\rho \sigma} - \Gamma^\sigma_{\mu \rho} \Gamma^\rho_{\nu \sigma} \]

with \[ \Gamma^\lambda_{\mu \nu} \approx O \left( \frac{1}{c^2} \right) \quad \frac{1}{c} \frac{\partial}{\partial t} \ll \frac{\partial}{\partial q^i} \]

\[ R_{00} \approx \Gamma^i_{00,i} \]

\[ \Gamma^i_{00} \approx -\frac{1}{2} g^{ij} g_{00,j} = -\frac{1}{c^2} g^{ij} \Phi_{,j} = -\frac{1}{c^2} \Phi_{,i} \quad g_{00} \approx 1 + \frac{2 \Phi}{c^2} \]

\[ R_{00} \approx -\frac{1}{c^2} \frac{\partial^2 \Phi}{\partial q^i \partial q_i} = \frac{1}{c^2} \nabla^2 \Phi \quad R_{00} \approx \frac{\kappa}{2} \rho c^2 \]

\[ \nabla^2 \Phi = \frac{\kappa}{2} \rho c^4 \]

Newton: \[ \nabla^2 \Phi = 4 \pi G \rho \]

\[ \kappa = \frac{8 \pi G}{c^4} \]
[note: Einstein’s original sign convention for the Ricci tensor differs from ours]

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6.3 Cosmological constant

modified Einstein tensor

\[ \tilde{G}^{\mu\nu} = G^{\mu\nu} + \Lambda g^{\mu\nu} \quad \Lambda = \text{const.} \]

also fulfills  \[ \tilde{G}^{\nu\mu} = \tilde{G}^{\mu\nu} \quad \tilde{G}^{\mu\nu}_{\ ,\nu} = 0 \]

measurements suggest  \[ \rho_\Lambda = \frac{c^2 \Lambda}{8\pi G} \simeq 10^{-29} \text{ g cm}^{-3} \]

Solar neighbourhood  \[ \rho_{\text{local}} \simeq 10^{-23} \text{ g cm}^{-3} \]

baryonic matter in the Universe  \[ \rho_\text{b} \simeq 5 \times 10^{-31} \text{ g cm}^{-3} \]

negligible correction, unless huge length scales are considered

\[
G^{\mu\nu} = \frac{8\pi G}{c^4} T^{\mu\nu} - \Lambda g^{\mu\nu}
\]

effective repulsion (dark) “vacuum” energy ??

theories modifying the law of gravity provide alternative models
6.4 Time and distance

Laws of physics — described by tensors — do not depend on coordinates

• coordinates do not have immediate physical meaning

• What is the time and distance?

\[ g_{\mu\nu} \] are not completely arbitrary

\[
ds^2 = g_{\mu\nu} \, dq^\mu \, dq^\nu
\]

can be locally transformed to

\[
ds^2 = c^2 \, dt^2 - d\mathbf{r}^2
\]

eigenvalues of matrix with

\[(g)_{ij} = g_{ij}
\]

have signs \((+1, -1, -1, -1)\)

corresponding to 1 time-like and 3 space-like coordinates

• \[ g = \det g < 0 \]
time interval $d\tau$ between two events at the same location given by
\[ ds^2 = g_{00} \left( dq^0 \right)^2 = c^2 d\tau^2 \]

\[ \tau = \frac{1}{c} \int \sqrt{g_{00}} \ dq^0 \] proper time

in general, the relation between the proper time interval depends on the location

cannot define spatial distance by means of for neighbouring events at the same time
GR-conform definition of (infinitesimal) spatial distance:

\[
time\text{ elapsed between emitting a light signal and receiving it back } \times \frac{c}{2}
\]

spatial metric

\[
dl^2 = \gamma_{ij} dq^i dq^j
\]

\[
\gamma_{ij} = -g_{ij} + \frac{g_{0i} g_{0j}}{g_{00}} \quad \Rightarrow \quad \gamma^{ij} = -g^{ij}
\]

\[
\gamma_{ij} = -g_{ij} \text{ holds for the case } g_{0i} = 0 \text{ only}
\]

definition of finite distance as \[\int dl\]

requires \[\gamma_{ij}\] not to depend on time
emit signal at point B with spatial coordinates \( q^i + dq^i \)
receive it at point A at \( q^i \) and send it back to B

Light ray has to fulfill

\[
\begin{align*}
\frac{ds^2}{g_{ij} dq^i dq^j} + 2 g_{0i} dq^0 dq^i + g_{00} (dq^0)^2 &= 0 \quad \text{\( \Leftrightarrow \)} \quad \frac{(dq^0)^2}{g_{00}} + 2 \frac{g_{0i}}{g_{00}} dq^0 dq^i + \frac{g_{ij}}{g_{00}} dq^i dq^j = 0 \\
\Rightarrow dq^0 &= \frac{-g_{0i}}{g_{00}} dq^i \pm \sqrt{\left(\frac{g_{0i}}{g_{00}} dq^i\right)^2 - \frac{g_{ij}}{g_{00}} dq^i dq^j} \\
&= \frac{1}{g_{00}} \left[-g_{0i} dq^i \pm \sqrt{(g_{0i} g_{0j} - g_{ij} g_{00}) dq^i dq^j}\right]
\end{align*}
\]

\[
\begin{align*}
\Rightarrow dl &= \frac{c}{2} d\tau = \frac{\sqrt{g_{00}}}{2} (dq^0_+ - dq^0_-) = \sqrt{\left(\frac{g_{0i} g_{0j}}{g_{00}} - g_{ij}\right) dq^i dq^j} = \sqrt{\gamma_{ij} dq^i dq^j} \\
\Rightarrow \gamma_{ij} &= -g_{ij} + \frac{g_{0i} g_{0j}}{g_{00}}
\end{align*}
\]
6.5 Synchronisation

exchange light signals between neighbouring points at infinitesimal distance

emit signal at point B with spatial coordinates \( q^i + dq^i \)
receive it at point A at \( q^i \), and send it back to B

synchronisation means shift in time coordinate

\[
\Delta q^0 = \frac{1}{2} \left( dq^0_+ + dq^0_- \right) = -\frac{g^{0i}}{g_{00}} dq^i
\]

along trajectory (and depends on it)

if \( g_{0i} = 0 \) \( \Rightarrow \Delta q^0 = 0 \)
if \( g_{0i} = 0 \) \( \Leftrightarrow \) global synchronisation possible
(with regard to time coordinate, but measured \( d\tau \) depends on location)

coordinate transformation can always provide
(at cost of time-dependent \( \gamma_{ij} \))

\[
\begin{array}{l}
\begin{aligned}
ds^2 &= c^2 dt^2 - \gamma_{ij} dq^i dq^j \\
(\text{synchronized reference frame})
\end{aligned}
\end{array}
\]

\[d\tau = dt \quad \text{everywhere}\]

coordinate line of \( t \) (i.e. \( q^i = \text{const.} \)) is geodesic