3 Classical Mechanics

3.1 Principle of stationary action

Mechanical system completely described by

$\mathcal{L}(q_i, \dot{q}_i, t)$ (Lagrangian)

q_i coordinate
$\dot{q}_i$ velocity
t time

action $S = \int \mathcal{L}(q_i, \dot{q}_i, t)\, \mathrm{d}t$

$\delta S = 0$ (Hamilton's) principle of stationary action

Fermat's principle (optics)
Feynman's path integral (QM)

↳ $\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = 0}$ (Euler-) Lagrange equations

for $\mathcal{L}(q_i, \dot{q}_i)$: $\mathcal{L} = T - V$

T kinetic energy
V potential energy

Example: 1D harmonic oscillator (I)

$$T = \frac{1}{2} m\dot{x}^2 \qquad V = \frac{1}{2} kx^2$$

$$\mathcal{L}(x, \dot{x}) = \frac{1}{2} m\dot{x}^2 - \frac{1}{2} kx^2$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) - \frac{\partial \mathcal{L}}{\partial x} = 0$$

↳ $$\frac{\mathrm{d}}{\mathrm{d}t}(m\dot{x}) + kx = 0$$

↳ $$\ddot{x} = -\frac{k}{m} x$$

↳ $$\begin{aligned} x(t) &= \mathfrak{Re}\left(A\, e^{\mathrm{i}\omega t} + B\, e^{-\mathrm{i}\omega t}\right) \\ &= a\, \cos(\omega t) + b\, \sin(\omega t) \\ &= \alpha\, \cos(\omega t - \varphi) \end{aligned} \qquad \omega = \sqrt{\frac{k}{m}}$$

3.2 Hamilton's equations

for each velocity $\dot{q}_i$

conjugate momentum $p_i = \dfrac{\partial \mathcal{L}}{\partial \dot{q}_i}$

$\mathcal{L}(q_i, \dot{q}_i, t) \quad \rightarrow \quad \mathcal{H}(q_i, p_i, t)$ (Hamiltonian)

by means of a Legendre transformation

$$\mathcal{H}(q_i, p_i, t) = \sum_j \dot{q}_j \, p_j - \mathcal{L}(q_i, \dot{q}_i, t)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = 0$$

↳ $$\boxed{\frac{\partial \mathcal{H}}{\partial q_i} = -\dot{p}_i \qquad \frac{\partial \mathcal{H}}{\partial p_i} = \dot{q}_i}$$ Hamilton's equations

for $\mathcal{H}(q_i, p_i)$: $\mathcal{H} = T + V$

Example: 1D harmonic oscillator (II)

$$\mathcal{L}(x, \dot{x}) = \frac{1}{2} m\dot{x}^2 - \frac{1}{2} kx^2$$

$$p = \frac{\partial \mathcal{L}}{\partial \dot{x}} = m\,\dot{x}$$

$$T = \frac{p^2}{2m} \qquad V = \frac{1}{2} kx^2$$

↳ $\mathcal{H}(x, p) = \frac{p^2}{2m} + \frac{1}{2} kx^2$

↳ $\frac{\partial \mathcal{H}}{\partial x} = kx = -\dot{p} \qquad \frac{\partial \mathcal{H}}{\partial p} = \frac{p}{m} = \dot{x}$

↳ $\dot{p} = m\ddot{x}$

↳ $\ddot{x} = -\frac{k}{m} x$

(as above...)

3.3 Hamilton-Jacobi formalism

action as function of q_i and t

$$S(q_i, t) = \int \mathcal{L}\,\mathrm{d}t = \int \left(\sum p_i\,\dot{q}_i - \mathcal{H} \right)\,\mathrm{d}t$$

↳ $\mathrm{d}S = \sum p_i\,\mathrm{d}q_i - \mathcal{H}\,\mathrm{d}t$ ↳ $\boxed{p_i = \dfrac{\partial S}{\partial q_i}}$

↳ $\boxed{\dfrac{\partial S}{\partial t} + \mathcal{H}\left(q_i, \dfrac{\partial S}{\partial q_i}, t\right) = 0}$ (Hamilton-Jacobi equation)

set of first order partial differential equations

Solution of the form $S = \tilde{S}(q_i, \alpha_i, t) + c$

with N constants α_i

for $\mathcal{H}\left(q_i, \frac{\partial S}{\partial q_i}, t\right) = \mathcal{H}\left(q_{k+1}, \ldots, q_N, \frac{\partial S}{\partial q_1} \ldots, \frac{\partial S}{\partial q_N}\right)$

↳ separation of the time and k coordinates q_i

with ansatz

$$S(q_i, t) = S_0(t) + \sum_{j=1}^{k} S_j(q_j) + \hat{S}(q_{k+1}, \ldots, q_N)$$

$$\frac{\partial S}{\partial t} = \frac{\partial S_0}{\partial t} = \alpha_0 \equiv -E \qquad \frac{\partial S}{\partial q_j} = \alpha_j \equiv p_j \quad (j = 1, \ldots, k)$$

↳ $$\boxed{S(q_i, \alpha_i, t) = \alpha_0\, t + \sum_{j=1}^{k} \alpha_j\, q_j + \hat{S}(q_{k+1}, \ldots, q_N, \alpha_0, \ldots, \alpha_{N-1})}$$

solve for $\hat{S}(q_{k+1}, \ldots, q_N)$ by means of Hamilton-Jacobi equation

equations of motion follow from $\boxed{\frac{\partial S}{\partial \alpha_i} = \text{const.}}$ $(i = 0, \ldots, N-1)$

Example: 1D harmonic oscillator (III)

$$\mathcal{H}\left(x, \frac{\partial S}{\partial x}\right) = \frac{1}{2m}\left(\frac{\partial S}{\partial x}\right)^2 + \frac{1}{2}kx^2 \qquad p_x = \frac{\partial S}{\partial x}$$

$$\mathcal{H}\left(x, \frac{\partial S}{\partial x}\right) + \frac{\partial S}{\partial t} = 0$$

$$\hookrightarrow \quad \frac{1}{2m}\left(\frac{\partial S}{\partial x}\right)^2 + \frac{1}{2}kx^2 + \frac{\partial S}{\partial t} = 0$$

$$S = -E\,t + S_x(x)$$

$$\hookrightarrow \quad \frac{1}{2m}\left(\frac{\partial S_x}{\partial x}\right)^2 + \frac{1}{2}kx^2 = E$$

$$\hookrightarrow \quad S_x = \int \sqrt{m\,(2E - kx^2)}\,\mathrm{d}x + C_0$$

$$\frac{\partial S}{\partial E} = -t + \int \sqrt{\frac{m}{2E - kx^2}}\,\mathrm{d}x = \text{const.}$$

$$\hookrightarrow \quad t = t_0 + \sqrt{\frac{m}{k}} \int \frac{\mathrm{d}x}{\sqrt{\frac{2E}{k} - x^2}} \qquad \int \frac{\mathrm{d}x}{\sqrt{a^2 - x^2}} = \arcsin\frac{x}{a}$$

$$t = t_0 + \omega^{-1} \arcsin\frac{x}{\sqrt{\frac{2E}{k}}} \qquad x(t) = \sqrt{\frac{k}{2E}}\,\sin[\omega(t - t_0)] \qquad \omega = \sqrt{\frac{k}{m}}$$

4 Mechanics in curved space

4.1 Equations of motion

free particle $\mathcal{L} = \frac{1}{2} m v^2 = \frac{1}{2} m \, g_{ij} \, \dot{q}^i \dot{q}^j$

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial \mathcal{L}}{\partial \dot{q}^l} \right) - \frac{\partial \mathcal{L}}{\partial q^l} = 0$$ (Euler-Lagrange equations)

$$\ddot{q}^k + \Gamma^k_{ij} \, \dot{q}^i \dot{q}^j = 0$$

equations of motion for free particle in curved space

$$\mathcal{L} = \frac{1}{2} m v^2 = \frac{1}{2} m\, g_{ij}\, \dot{q}^i \dot{q}^j \qquad v^2 = \left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 = g_{ij}\, \dot{q}^i \dot{q}^j \qquad \mathrm{d}\boldsymbol{s} = \boldsymbol{e}_i\, \mathrm{d}q^i$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}^l}\right) - \frac{\partial \mathcal{L}}{\partial q^l} = 0 \qquad \text{(Euler-Lagrange equations)}$$

$$\hookrightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(m\, g_{il}\, \dot{q}^i\right) - \frac{1}{2} m \frac{\partial g_{ij}}{\partial q^l} \dot{q}^i \dot{q}^j = 0$$

$$\hookrightarrow \quad \left(\frac{\mathrm{d}}{\mathrm{d}t} g_{il}\right) \dot{q}^i + g_{il}\, \ddot{q}^i - \frac{1}{2} \frac{\partial g_{ij}}{\partial q^l} \dot{q}^i \dot{q}^j = 0$$

$$\left.\begin{array}{r} \\ \\ \\ \end{array}\right\} \qquad \frac{\partial g_{il}}{\partial q^j} \dot{q}^i \dot{q}^j = \frac{1}{2}\left(\frac{\partial g_{il}}{\partial q^j} + \frac{\partial g_{jl}}{\partial q^i}\right) \dot{q}^i \dot{q}^j$$

$$\hookrightarrow \quad g_{il}\, \ddot{q}^i + \frac{1}{2}\left(\frac{\partial g_{il}}{\partial q^j} + \frac{\partial g_{jl}}{\partial q^i} - \frac{\partial g_{ij}}{\partial q^l}\right) \dot{q}^i \dot{q}^j = 0$$

with $\quad g_{il}\, g^{lk} = \delta_i{}^k \quad$ and $\quad \Gamma^k_{ij} \equiv \frac{1}{2}\, g^{lk}\left(\frac{\partial g_{il}}{\partial q^j} + \frac{\partial g_{jl}}{\partial q^i} - \frac{\partial g_{ij}}{\partial q^l}\right)$

$$\hookrightarrow \quad \boxed{\ddot{q}^k + \Gamma^k_{ij}\, \dot{q}^i \dot{q}^j = 0}$$

4.2 Embedding

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}y^2 = \mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 + r^2\,\sin^2\theta\,\mathrm{d}\varphi^2$$

describes flat 3-dimensional space

(curved) surface embedded in 3-dimensional space

$$\mathrm{d}s^2 = g_{ij}\,\mathrm{d}q^i\,\mathrm{d}q^j \qquad (N=3)$$

defined by $q^3 = \mathrm{const.}$

$$\mathrm{d}s^2 = g_{ij}\,\mathrm{d}q^i\,\mathrm{d}q^j \qquad (N=2)$$

e.g. spherical surface defined by $r = \mathrm{const.}$

$$ds^2 = r^2\,\mathrm{d}\theta^2 + r^2\,\sin^2\theta\,\mathrm{d}\varphi^2$$

with $q^3 = 0$

motion restricted to (curved) surface determined by

$$\ddot{q}^k + \Gamma^k_{ij}\,\dot{q}^i\dot{q}^j = 0 \qquad (N=2)$$

(unaffected by the presence of further dimensions)

4.3 Geodesics

$\frac{\mathrm{d}\boldsymbol{s}}{\mathrm{d}\lambda}$ tangent vector to a curve $C(\lambda)$

$C(\lambda)$ is geodesic if its tangent vector remains parallel to itself i.e.

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\mathrm{d}\boldsymbol{s}}{\mathrm{d}\lambda}\right) = 0}$$

- moving along geodesics means to keep the same direction
- geodesics form straight lines in the sequence of connected tangent spaces

$$\Leftrightarrow \quad \boxed{\frac{\mathrm{d}^2 q^k}{\mathrm{d}\lambda^2} + \Gamma^k_{ij}\,\frac{\mathrm{d}q^i}{\mathrm{d}\lambda}\,\frac{\mathrm{d}q^j}{\mathrm{d}\lambda} = 0} \quad \text{(geodesic equation)}$$

4.4 Stationary paths

$$\int_C \mathrm{d}s = \int_{\lambda_0}^{\lambda_1} \left| \frac{\mathrm{d}\boldsymbol{s}[q(\lambda)]}{\mathrm{d}\lambda} \right| \mathrm{d}\lambda = \int_{\lambda_0}^{\lambda_1} \sqrt{g_{ij}[q(\lambda)] \frac{\mathrm{d}q^i}{\mathrm{d}\lambda} \frac{\mathrm{d}q^j}{\mathrm{d}\lambda}} \, \mathrm{d}\lambda$$

$$\boxed{\delta \int \mathrm{d}s = 0}$$ stationary path between two points (e.g. path length is locally shortest)

$$\Leftrightarrow \quad \boxed{\frac{\mathrm{d}^2 q^k}{\mathrm{d}\lambda^2} + \Gamma^k_{ij} \frac{\mathrm{d}q^i}{\mathrm{d}\lambda} \frac{\mathrm{d}q^j}{\mathrm{d}\lambda} = 0} \quad \text{(geodesic equation)}$$

stationary path and geodesic equation

$$\delta \int \mathrm{d}s = \delta \int \sqrt{g_{ij}[q(\lambda)] \frac{\mathrm{d}q^i}{\mathrm{d}\lambda} \frac{\mathrm{d}q^j}{\mathrm{d}\lambda}}\, \mathrm{d}\lambda = 0$$

with $\mathcal{L}\left(q^i, \frac{\mathrm{d}q^i}{\mathrm{d}\lambda}\right) = \frac{1}{2} m\, g_{ij}[q(\lambda)] \frac{\mathrm{d}q^i}{\mathrm{d}\lambda} \frac{\mathrm{d}q^j}{\mathrm{d}\lambda}$

$$\hookrightarrow \quad \mathcal{G}\left(q^i, \frac{\mathrm{d}q^i}{\mathrm{d}\lambda}\right) = \sqrt{g_{ij}[q(\lambda)] \frac{\mathrm{d}q^i}{\mathrm{d}\lambda} \frac{\mathrm{d}q^j}{\mathrm{d}\lambda}} = \left[\frac{2}{m} \mathcal{L}\left(q^i, \frac{\mathrm{d}q^i}{\mathrm{d}\lambda}\right)\right]^{1/2}$$

$$\delta \int \mathcal{G}\left(q^i, \frac{\mathrm{d}q^i}{\mathrm{d}\lambda}\right) \mathrm{d}\lambda = 0$$

$$\hookrightarrow \quad \frac{\mathrm{d}}{\mathrm{d}\lambda}\left[\frac{\partial \mathcal{G}}{\partial\left(\mathrm{d}q^k/\mathrm{d}\lambda\right)}\right] - \frac{\partial \mathcal{G}}{\partial q^k} = 0 \quad \Leftrightarrow \quad \frac{1}{\sqrt{2m\mathcal{L}}}\left\{\frac{\mathrm{d}}{\mathrm{d}\lambda}\left[\frac{\partial \mathcal{L}}{\partial\left(\mathrm{d}q^k/\mathrm{d}\lambda\right)}\right] - \frac{\partial \mathcal{L}}{\partial q^k}\right\} = 0$$

equivalence to principle of stationary action $\quad \frac{\mathrm{d}^2 q^k}{\mathrm{d}\lambda^2} + \Gamma^k_{ij} \frac{\mathrm{d}q^i}{\mathrm{d}\lambda} \frac{\mathrm{d}q^j}{\mathrm{d}\lambda} = 0$

$$\delta \int \mathcal{L}\,\mathrm{d}t = 0$$

based on Newton's law

$$\delta \int \mathrm{d}s = 0$$

purely space geometry

$$\boxed{\frac{\mathrm{d}^2 q^k}{\mathrm{d}\lambda^2} + \Gamma^k_{ij}\,\frac{\mathrm{d}q^i}{\mathrm{d}\lambda}\,\frac{\mathrm{d}q^j}{\mathrm{d}\lambda} = 0}$$

(motion along geodesics)

$$\Leftrightarrow \quad \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\mathrm{d}\boldsymbol{s}}{\mathrm{d}\lambda}\right) = 0$$

4.5 Conserved quantities

$$\frac{\mathrm{d}^2 q^k}{\mathrm{d}\lambda^2} + \Gamma^k_{ij} \frac{\mathrm{d}q^i}{\mathrm{d}\lambda} \frac{\mathrm{d}q^j}{\mathrm{d}\lambda} = 0 \quad \Leftrightarrow \quad \boxed{\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(g_{kl} \frac{\mathrm{d}q^l}{\mathrm{d}\lambda} \right) = \frac{1}{2} g_{ij,k} \frac{\mathrm{d}q^i}{\mathrm{d}\lambda} \frac{\mathrm{d}q^j}{\mathrm{d}\lambda}}$$

(geodesic equation)

if all g_{ij} do not depend on q^k $\quad \hookrightarrow \quad \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(g_{kl} \frac{\mathrm{d}q^l}{\mathrm{d}\lambda} \right) = 0$

$$\mathcal{L} = \frac{1}{2}\, m\, g_{ij}(q)\, \dot{q}^i\, \dot{q}^j \qquad \mathcal{H} = \frac{1}{2m}\, g^{ij}(q)\, p_i\, p_j \quad (= \mathcal{L})$$

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pk being the conjugate momentum of the coordinate q^k

$$\ddot{q}^k + \Gamma^k_{ij}\, \dot{q}^i \dot{q}^j = 0 \quad \Leftrightarrow \quad \left(g_{kl}\, \dot{q}^l\right)^{\cdot} = \frac{1}{2}\, g_{ij,k}\, \dot{q}^i\, \dot{q}^j$$

$$\Leftrightarrow \quad \dot{p}_k = -\frac{\partial \mathcal{H}}{\partial q^k}$$

(↔ Hamilton's equations)

$g_{ij,k} = 0$ for all i and j $\Rightarrow p_k = \text{const.} \quad \not\Rightarrow p^k = m\, \dot{q}^k = \text{const.}$

4.6 Hamilton-Jacobi equation

$$\frac{\partial S}{\partial t} + \mathcal{H}\left(q^i, \frac{\partial S}{\partial q^i}\right) = 0 \qquad \mathcal{H}\left(q^i, p_i\right) = \frac{1}{2m} g^{ij}(q)\, p_i\, p_j$$

$$\boxed{\frac{\partial S}{\partial t} + \frac{1}{2m} g^{ij}(q)\, \frac{\partial S}{\partial q^i} \frac{\partial S}{\partial q^j} = 0}$$

first-order differential equation
depends on metric tensor,
but not on its derivatives

$$E = -\frac{\partial S}{\partial t} \qquad p_k = \frac{\partial S}{\partial q^k} \qquad \Rightarrow \quad E = \frac{\boldsymbol{p}^2}{2m}$$

corresponds to energy-momentum relation

all g^{ij} do not depend on q^k $\quad \Rightarrow \quad \frac{\partial S}{\partial q^k} = p_k = \text{const.}$

identification of conserved quantities hard-wired into formalism

$$\frac{\partial S}{\partial t} + \frac{1}{2m}\, g^{ij}(q)\, \frac{\partial S}{\partial q^i}\, \frac{\partial S}{\partial q^j} = 0$$

$$E = -\frac{\partial S}{\partial t} \qquad\qquad p_k = \frac{\partial S}{\partial q^k}$$

Reminded of something ?

in quantum mechanics:

$$E \to \mathrm{i}\,\hbar\,\frac{\partial}{\partial t} \qquad\qquad p_k \to -\mathrm{i}\,\hbar\,\frac{\partial}{\partial q^k}$$

$$E = \frac{\boldsymbol{p}^2}{2m} \qquad \hookrightarrow \quad \mathrm{i}\,\hbar\,\frac{\partial}{\partial t}\,\psi(\boldsymbol{r},t) = -\frac{\hbar^2}{2m}\,\boldsymbol{\nabla}^2\,\psi(\boldsymbol{r},t)$$

(Schrödinger equation)

5 Special Relativity

5.1 Minkowski space

"event" described by time and location $(t, \boldsymbol{r})$
"reference frame"
  defines coordinate origin and motion
"inertial system"
  force-free particles move uniformly

all reference frames moving uniformly with respect to an inertial system
  are inertial system themselves
laws of physics assume the same form in all inertial systems

invariance of speed of light

- along light rays:

$$c\,\mathrm{d}t = |\mathrm{d}\boldsymbol{r}| \quad \text{for all reference frames}$$

- $\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 - \mathrm{d}\boldsymbol{r}^2 = 0$

homogeneity and isotropy of space and time

- invariance of $\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 - \mathrm{d}\boldsymbol{r}^2$
  - $\mathrm{d}s$

$\mathrm{d}t$ $\mathrm{d}\boldsymbol{r}$

ds describes distance in four-dimensional space
whereas both dt and dr depend on reference frame

use 4-dimensional vectors

$$\mathrm{d}q^\mu = (c\,\mathrm{d}t, \mathrm{d}q^i) \qquad \mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}q^\mu\,\mathrm{d}q^\nu$$

Latin indices $\in \{1,2,3\}$
Greek indices $\in \{0,1,2,3\}$

flat three-dimensional space described by cartesian coordinates

$$\mathrm{d}\boldsymbol{r}^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$$

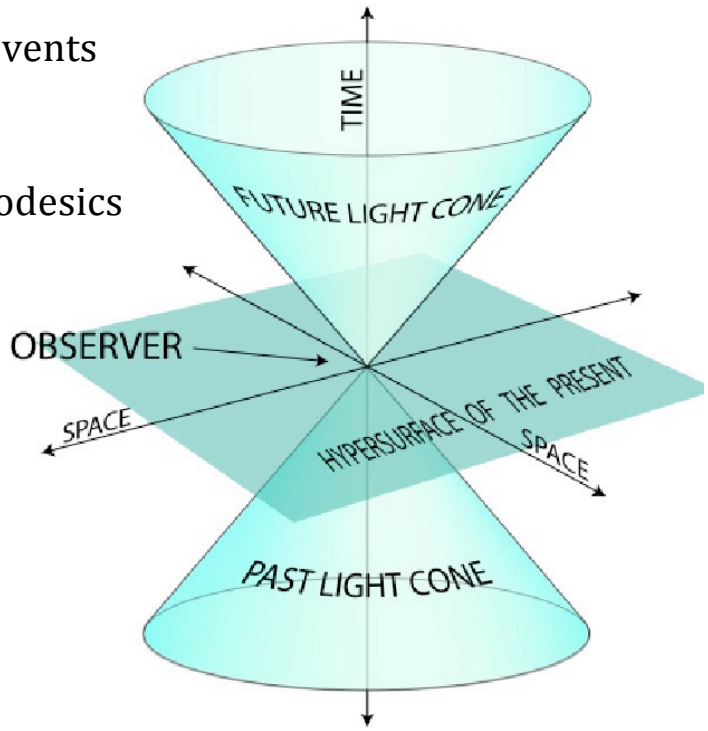
$$\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 - \mathrm{d}\boldsymbol{r}^2 \quad \hookrightarrow \quad \mathbf{g} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

5.2 Light cone

or: “causality and the finite speed of light”

$ds^2 = 0$ defines light cone

45° opening angle in (ct, x^i)



photons trace null geodesics between events

$ds^2 > 0$ inside light cone

massive particles move on time-like geodesics

$ds^2 < 0$ outside light cone

‘elsewhere’, no causal connection

non-relativistic theories:

instantaneous knowledge of interaction

$c \to \infty$ light cone widens,

all events get into causal contact

invariance of ds^2

↳ categorization holds irrespective of coordinate system and reference frame

5.3 Proper time

along worldline $q^\mu(\lambda)$ of clock with attached rest frame $(\tau, \hat{x}, \hat{y}, \hat{z})$

$$\mathrm{d}s^2 = c^2\,\mathrm{d}\tau^2$$

$$\hookrightarrow \quad \boxed{\mathrm{d}\tau = \frac{\mathrm{d}s}{c} \qquad \text{proper time}}$$

time shown on clock $\displaystyle\int \mathrm{d}\tau = \frac{1}{c}\int \mathrm{d}s$

invariance of $\mathrm{d}s^2$ $\hookrightarrow$ $c^2\,\mathrm{d}\tau^2 = c^2\,\mathrm{d}t^2 - \mathrm{d}\boldsymbol{r}^2$

$$\hookrightarrow \quad \mathrm{d}\tau = \sqrt{1 - \frac{v^2}{c^2}}\;\mathrm{d}t \qquad\qquad \boldsymbol{v} = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}$$

so that $\mathrm{d}\tau \leqslant \mathrm{d}t$ (moving clock is observed lacking behind)

5.4 Relativistic mechanics

define 4-velocity as

$$u^\mu = c\,\frac{\mathrm{d}q^\mu}{\mathrm{d}s} \qquad\qquad \mathrm{d}q^\mu = (c\,\mathrm{d}t, \mathrm{d}q^i)$$

$$\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 - \mathrm{d}\boldsymbol{r}^2 = \left(c^2 - \boldsymbol{v}^2\right)\,\mathrm{d}t^2 \quad \hookrightarrow \quad \frac{\mathrm{d}s}{c} = \sqrt{1 - \frac{v^2}{c^2}}\;\mathrm{d}t$$

$$\hookrightarrow \quad u^\mu = \frac{(c, v^i)}{\sqrt{1 - \frac{v^2}{c^2}}}$$

known: free particle moves along geodesic

$$\hookrightarrow \quad \frac{\mathrm{d}u^\mu}{\mathrm{d}s} = 0 \qquad [\text{ all } \Gamma^\lambda_{\mu\nu} = 0\,] \qquad\qquad \hookrightarrow \quad v^i = \text{const.}$$

(as anticipated for inertial system)

relativistic action

ansatz: $S = \int \mathcal{L}\,\mathrm{d}t = \alpha \int \mathrm{d}s$ (matches invariance of $\mathrm{d}s$)

↳ $\mathcal{L} = \alpha\, c \sqrt{1 - \dfrac{v^2}{c^2}}$

non-relativistic limit ($c \to \infty$)

$$\mathcal{L} \simeq \alpha\, c \left(1 - \frac{1}{2}\frac{v^2}{c^2}\right) = \alpha\, c - \frac{\alpha}{2c}\, v^2 \qquad \mathcal{L} = \frac{1}{2} m\, v^2 + \text{const.}$$

↳ $\alpha = -m\, c$ ↳

$$\boxed{\mathcal{L} = -m\, c^2 \sqrt{1 - \frac{v^2}{c^2}} \qquad S = -m\, c \int \mathrm{d}s}$$

conjugate momentum

$$p_k = \frac{\partial \mathcal{L}}{\partial \dot{q}^k} = \frac{m\,\dot{q}_k}{\sqrt{1-\frac{v^2}{c^2}}}$$

energy

$$E = \mathcal{H} = p_i\,\dot{q}^i - \mathcal{L} = \frac{m\,c^2}{\sqrt{1-\frac{v^2}{c^2}}}$$

↳ $$\left(\frac{E}{c}\right)^2 - \boldsymbol{p}^2 = m^2\,c^2$$

$$p^\mu = \left(E/c, p^i\right) \qquad p^\mu = m\,u^\mu$$

with $p_\mu = -\dfrac{\partial S}{\partial q^\mu}$ (sign in spatial part due to $-\mathrm{d}\boldsymbol{r}^2$ in metric)

$$p_\mu\,p^\mu = m^2\,c^2 \quad \hookrightarrow \quad \boxed{g^{\mu\nu}\,\frac{\partial S}{\partial q^\mu}\,\frac{\partial S}{\partial q^\nu} - m^2\,c^2 = 0}$$

(relativistic Hamilton-Jacobi equation)