1. $L = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$.

$$\therefore \quad \frac{\partial L}{\partial\left(\frac{dy}{dx}\right)} = \frac{dy/dx}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}$$

Since $\frac{\partial L}{\partial y} = 0$, the solution of

$$\frac{d}{dx} \frac{\partial L}{\partial\left(\frac{dy}{dx}\right)} - \frac{\partial L}{\partial y} = 0 \quad \text{is}$$

$$\frac{\partial L}{\partial\left(\frac{dy}{dx}\right)} = \text{const} = k \text{ say.}$$

$$\therefore \quad \left(\frac{dy}{dx}\right)^2 = k^2\left[1 + \left(\frac{dy}{dx}\right)^2\right]$$

$$\therefore \quad \left(\frac{dy}{dx}\right)^2 = \frac{k^2}{1-k^2}$$

$$\therefore \quad \frac{dy}{dx} = \pm \frac{k}{\sqrt{1-k^2}} = m \text{ say}$$

$$\Rightarrow \quad y = mx + c.$$

2.

$$\Gamma^{\lambda}_{\mu\nu} = \tfrac{1}{2} g^{\lambda\sigma} \left(g_{\mu\sigma,\nu} + g_{\nu\sigma,\mu} - g_{\mu\nu,\sigma} \right)$$

so $$\Gamma^{\lambda}_{\kappa\nu} = \tfrac{1}{2} g^{\lambda\sigma} \left(g_{\kappa\sigma,\nu} + g_{\nu\sigma,\kappa} - g_{\kappa\nu,\sigma} \right)$$

$$\therefore\ g_{\lambda\kappa} \Gamma^{\lambda}_{\mu\nu} + g_{\lambda\mu} \Gamma^{\lambda}_{\kappa\nu} = \tfrac{1}{2} g_{\lambda\kappa} g^{\lambda\sigma} \left(g_{\mu\sigma,\nu} + g_{\nu\sigma,\mu} - g_{\mu\nu,\sigma} \right)$$
$$+ \tfrac{1}{2} g_{\lambda\mu} g^{\lambda\sigma} \left(g_{\kappa\sigma,\nu} + g_{\nu\sigma,\kappa} - g_{\kappa\nu,\sigma} \right)$$

But $g_{\lambda\kappa} g^{\lambda\sigma} = \delta^{\sigma}_{\kappa}$, $g_{\lambda\mu} g^{\lambda\sigma} = \delta^{\sigma}_{\mu}$

$\therefore$ carrying out in each case the sum with respect to σ,

$$g_{\lambda\kappa} \Gamma^{\lambda}_{\mu\nu} + g_{\lambda\mu} \Gamma^{\lambda}_{\kappa\nu} = \tfrac{1}{2} \left(g_{\mu\kappa,\nu} + \cancel{g_{\nu\kappa,\mu}} - \cancel{g_{\mu\nu,\kappa}} \right)$$
$$+ \tfrac{1}{2} \left(g_{\kappa\mu,\nu} + \cancel{g_{\nu\mu,\kappa}} - \cancel{g_{\kappa\nu,\mu}} \right)$$

$$= g_{\mu\kappa,\nu}$$

taking account of the symmetry of the metric tensor.

3. $\Gamma'^{\kappa}_{\mu\nu} = \frac{1}{2} g'^{\kappa\lambda} \left(g'_{\mu\lambda,\nu} + g'_{\nu\lambda,\mu} - g'_{\mu\nu,\lambda} \right)$

Now $g'_{\mu\lambda,\nu} = \frac{\partial}{\partial x'^{\nu}} \left(\frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial x^{\beta}}{\partial x'^{\lambda}} g_{\alpha\beta} \right)$

Noting that $\frac{\partial}{\partial x'^{\nu}} g_{\alpha\beta} = \frac{\partial x^{\gamma}}{\partial x'^{\nu}} \frac{\partial}{\partial x^{\gamma}} g_{\alpha\beta} = \frac{\partial x^{\gamma}}{\partial x'^{\nu}} g_{\alpha\beta,\gamma}$

we have

$$g'_{\mu\lambda,\nu} = \frac{\partial^2 x^{\alpha}}{\partial x'^{\nu} \partial x'^{\mu}} \frac{\partial x^{\beta}}{\partial x'^{\lambda}} g_{\alpha\beta} + \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial^2 x^{\beta}}{\partial x'^{\nu} \partial x'^{\lambda}} g_{\alpha\beta}$$
$$+ \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial x^{\beta}}{\partial x'^{\lambda}} \frac{\partial x^{\gamma}}{\partial x'^{\nu}} g_{\alpha\beta,\gamma}.$$

$$\therefore g'_{\mu\lambda,\nu} + g'_{\nu\lambda,\mu} - g'_{\mu\nu,\lambda} = \frac{\partial^2 x^{\alpha}}{\partial x'^{\nu} \partial x'^{\mu}} \frac{\partial x^{\beta}}{\partial x'^{\lambda}} g_{\alpha\beta} + \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial^2 x^{\beta}}{\partial x'^{\nu} \partial x'^{\lambda}} g_{\alpha\beta}$$
$$+ \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial x^{\beta}}{\partial x'^{\lambda}} \frac{\partial x^{\gamma}}{\partial x'^{\nu}} g_{\alpha\beta,\gamma} + \frac{\partial^2 x^{\alpha}}{\partial x'^{\mu} \partial x'^{\nu}} \frac{\partial x^{\beta}}{\partial x'^{\lambda}} g_{\alpha\beta} + \frac{\partial x^{\alpha}}{\partial x'^{\nu}} \frac{\partial^2 x^{\beta}}{\partial x'^{\mu} \partial x'^{\lambda}} g_{\alpha\beta}$$
$$+ \frac{\partial x^{\alpha}}{\partial x'^{\nu}} \frac{\partial x^{\beta}}{\partial x'^{\lambda}} \frac{\partial x^{\gamma}}{\partial x'^{\mu}} g_{\alpha\beta,\gamma} - \frac{\partial^2 x^{\alpha}}{\partial x'^{\lambda} \partial x'^{\mu}} \frac{\partial x^{\beta}}{\partial x'^{\nu}} g_{\alpha\beta}$$
$$- \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial^2 x^{\beta}}{\partial x'^{\lambda} \partial x'^{\nu}} g_{\alpha\beta} - \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial x^{\beta}}{\partial x'^{\nu}} \frac{\partial x^{\gamma}}{\partial x'^{\lambda}} g_{\alpha\beta,\gamma}$$

(cancel, on interchanging α and β in one of them)

$$= 2 \frac{\partial^2 x^{\alpha}}{\partial x'^{\nu} \partial x'^{\mu}} \frac{\partial x^{\beta}}{\partial x'^{\lambda}} g_{\alpha\beta} + \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial x^{\beta}}{\partial x'^{\nu}} \frac{\partial x^{\gamma}}{\partial x'^{\lambda}} \left(g_{\alpha\gamma,\beta} + g_{\beta\gamma,\alpha} - g_{\alpha\beta,\gamma} \right)$$

noting that $\frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial x^{\beta}}{\partial x'^{\lambda}} \frac{\partial x^{\gamma}}{\partial x'^{\nu}} g_{\alpha\beta,\gamma} = \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial x^{\gamma}}{\partial x'^{\lambda}} \frac{\partial x^{\beta}}{\partial x'^{\nu}} g_{\alpha\gamma,\beta}$

on interchange of the dummy suffices β and γ, and similarly in one of the other terms.

3 cont.

Hence

$$\Gamma'^{\kappa}_{\mu\nu} = \frac{\partial x'^{\kappa}}{\partial x^{\rho}} \frac{\partial x'^{\lambda}}{\partial x^{\sigma}} g^{\rho\sigma} \Big[\frac{\partial^2 x^{\alpha}}{\partial x'^{\nu} \partial x'^{\mu}} \frac{\partial x^{\beta}}{\partial x'^{\lambda}} g_{\alpha\beta}$$

$$+ \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial x^{\beta}}{\partial x'^{\nu}} \frac{\partial x^{\gamma}}{\partial x'^{\lambda}} \tfrac{1}{2}\left(g_{\alpha\gamma,\beta} + g_{\beta\gamma,\alpha} - g_{\alpha\beta,\gamma} \right) \Big]$$

Now $\frac{\partial x'^{\lambda}}{\partial x^{\sigma}} \frac{\partial x^{\beta}}{\partial x'^{\lambda}} = \frac{\partial x^{\beta}}{\partial x^{\sigma}} = \delta^{\beta}_{\sigma}$

and $\frac{\partial x'^{\lambda}}{\partial x^{\sigma}} \frac{\partial x^{\gamma}}{\partial x'^{\lambda}} = \frac{\partial x^{\gamma}}{\partial x^{\sigma}} = \delta^{\gamma}_{\sigma}$

$$\therefore \Gamma'^{\kappa}_{\mu\nu} = \frac{\partial x'^{\kappa}}{\partial x^{\rho}} \underbrace{g^{\rho\beta} g_{\alpha\beta}}_{\delta^{\rho}_{\alpha}} \frac{\partial^2 x^{\alpha}}{\partial x'^{\mu} \partial x'^{\nu}}$$

$$+ \frac{\partial x'^{\kappa}}{\partial x^{\rho}} g^{\rho\gamma} \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial x^{\beta}}{\partial x'^{\nu}} \times \tfrac{1}{2}\left(g_{\alpha\gamma,\beta} + g_{\beta\gamma,\alpha} - g_{\alpha\beta,\gamma} \right)$$

$$= \frac{\partial x'^{\kappa}}{\partial x^{\rho}} \frac{\partial^2 x^{\rho}}{\partial x'^{\mu} \partial x'^{\nu}} + \frac{\partial x'^{\kappa}}{\partial x^{\rho}} \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial x^{\beta}}{\partial x'^{\nu}} \Gamma^{\rho}_{\alpha\beta}$$

4 $\quad g_{11} = a^2 \quad g_{22} = a^2 \sin^2\theta \quad g_{12} = g_{21} = 0.$

The only non-zero derivative is $\quad g_{22,1} = \frac{\partial}{\partial\theta} a^2 \sin^2\theta$

$= 2a^2 \sin\theta \cos\theta.$

Now $\quad \Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\kappa} (g_{\mu\kappa,\nu} + g_{\nu\kappa,\mu} - g_{\mu\nu,\kappa})$

Since $g_{\mu\nu}$ is diagonal, so also is $g^{\mu\nu}$ and the diagonal elements of the latter are the reciprocal of the diagonal elements of the former.

So $\quad g^{11} = \frac{1}{a^2} \quad g^{22} = \frac{1}{a^2\sin^2\theta}$

$\therefore \Gamma^{1}_{\mu\nu} = \frac{1}{2} g^{1\kappa} (g_{\mu\kappa,\nu} + g_{\nu\kappa,\mu} - g_{\mu\nu,\kappa})$

$= \frac{1}{2} g^{11} (g_{\mu1,\nu} + g_{\nu1,\mu} - g_{\mu\nu,1}) \quad$ since $g^{12} = 0$

$= \frac{1}{2a^2} (\underset{0}{g_{\mu1,\nu}} + \underset{0}{g_{\nu1,\mu}} - g_{\mu\nu,1})$

since only g_{22} has a non-zero derivative.

$\therefore \Gamma^{1}_{22} = -\frac{1}{2a^2} 2a^2 \sin\theta\cos\theta = -\sin\theta\cos\theta$

all others zero with superscript = 1.

Similarly $\quad \Gamma^{2}_{\mu\nu} = \frac{1}{2} g^{22} (g_{\mu2,\nu} + g_{\nu2,\mu} - \underset{0}{g_{\mu\nu,2}})$

$= \frac{1}{2a^2\sin^2\theta} (g_{\mu2,\nu} + g_{\nu2,\mu})$

$\therefore \Gamma^{2}_{12} = \Gamma^{2}_{21} = \frac{1}{2a^2\sin^2\theta} g_{22,1} = \cot\theta.$

General Relativity

Problem Sheet 1

1. Consider a third order mixed tensor T^p_{qr} where the suffices p, q and r take on the possible values 1 and 2. Write down systematically all components of this tensor, and the components of the tensors formed by *contraction*, i.e. T^p_{pr} and T^p_{qp}.

2. Evaluate the contracted form of the Kronecker delta, δ^p_p, in n dimensions.

3. Show that if A^{rs} is an antisymmetric contravariant tensor and B_{rs} is a symmetric covariant tensor, then
$$A^{rs}B_{rs} = 0.$$

4. On a Euclidean plane, the position of a point may be specified by cartesian coordinates (x,y) or by plane polar coordinates (r, θ) which are related by
$$x = r\cos\theta, \quad y = r\sin\theta$$
Consider now the distance $d\ell$ between two neighbouring points (r, θ) and $(r+dr, \theta+d\theta)$. Using the relations
$$dx = \frac{\partial x}{\partial r}dr + \frac{\partial x}{\partial \theta}d\theta, \quad dy = \frac{\partial y}{\partial r}dr + \frac{\partial y}{\partial \theta}d\theta,$$
evaluate the partial derivatives $\frac{\partial x}{\partial r}$, $\frac{\partial x}{\partial \theta}$ etc. and hence by substitution show that
$d\ell^2 = dx^2 + dy^2$ can be expressed in polar coordinates as $d\ell^2 = dr^2 + r^2 d\theta^2$.
Show by means of a diagram how this relation may be verified geometrically.

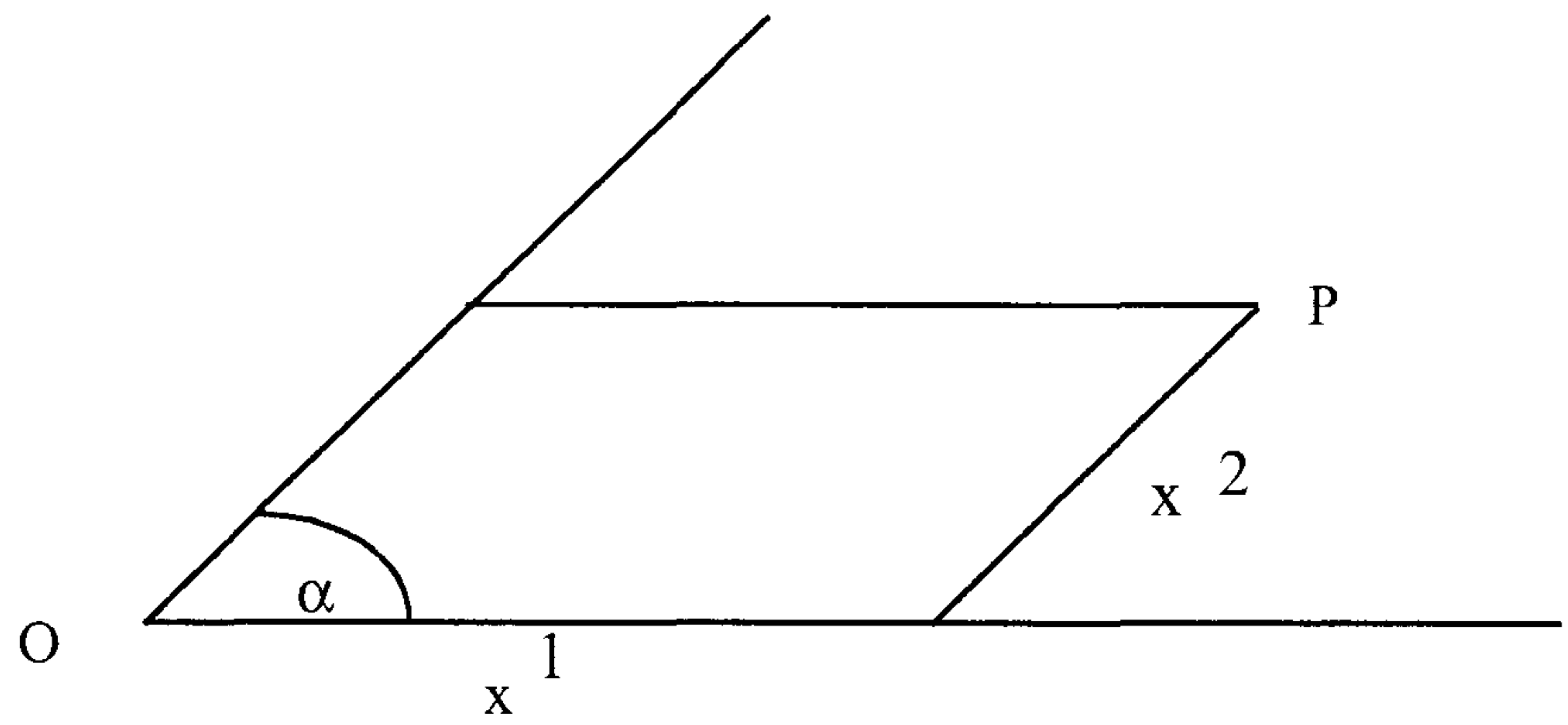
5. Show that if T_{mn} is a symmetric covariant tensor, so also is its contravariant counterpart T^{mn}, and that there is no distinction between $T^m{}_n$ and $T_n{}^m$.

6. Show that if the metric tensor is diagonal, each diagonal element satisfies $g_{nn} = 1/g^{nn}$.

7. Verify the identity $g^{mn}\frac{\partial}{\partial x^r}g_{mn} = \frac{\partial}{\partial x^r}\ln|g|$ for the case of a Euclidean plane, using plane polar coordinates (r, θ).

8. Again in the Euclidean plane, let the undashed coordinates x^r refer to cartesian coordinates (x,y), and dashed coordinates x'^r refer to plane polar coordinates (r, θ). If the vector field X at a certain point has components $X^r = (1,0)$, find the components X'^r at that point. Check that $g_{mn}X^m X^n$ has the same value in the two coordinate systems.

9. Consider a plane flat surface, in which we draw two axes which intersect at an angle α at a point O. Choosing any point P in the plane, the coordinates x^r (r=1,2) of P - i.e. the contravariant components of the vector OP - can be defined by drawing lines through P *parallel* to the axes and measuring the distances between O and the points of intersection with the axes, as shown in the diagram.



By considering a small displacement dx^r from P (coordinates x^r) to Q (coordinates x^r+dx^r) , show that the square of the distance between P and Q is of the form

$$d\ell^2 = g_{rs}dx^r dx^s$$

where $g_{11} = g_{22} = 1$, and $g_{12} = g_{21} = \cos\alpha$. (Hint: apply the cosine rule.) Hence by lowering the suffix with the aid of the metric tensor, show that the *covariant* components x_r (r =1,2) of the vector OP are given by

$$x_1 = x^1 + x^2 \cos\alpha, \quad x_2 = x^2 + x^1 \cos\alpha,$$

and that the latter components are obtained geometrically by measuring the distances from O to the points of intersection (with the axes) of lines drawn from P *perpendicular* to the axes.

[Note: we could of course have defined the coordinates at the outset the other way round, i.e. taking the contravariant components x^μ to be the numbers generated by dropping lines *perpendicular* to the axes, and later identifying the covariant components by the alternative construction involving lines *parallel* to the axes. The choice is arbitrary.Whichever set we define as the contravariant components, the other set form the covariant components. There is of course no distinction between covariant and contravariant components when the two axes are at right angles.

GENERAL RELATIVITY

Problem sheet 2

1. Show that in a two dimensional flat space, the condition for a path between two fixed points to have minimum length, i.e.

$$\delta\int d\ell = 0,$$

is that the path should take the form of a straight line, i.e. y = mx+c.

Hint: begin with the simple relation $d\ell = \sqrt{dx^2 + dy^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = L dx$

where

$$L = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

You may assume that the variational condition given above, which may now be written as

$$\delta\int L dx = 0,$$

with the end points of the path understood to be fixed, is equivalent to the analogue of Lagrange's equation,

$$\frac{d}{dx}\left(\frac{\partial L}{\partial \frac{dy}{dx}}\right) - \frac{\partial L}{\partial y} = 0$$

where, in our case, L is clearly a function of dy/dx only. Calculate the appropriate derivative of L and hence solve for the relation between y and x.

2. Show that the partial derivative of the metric tensor may be expressed as

$$g_{\mu\kappa,\nu} = g_{\lambda\kappa}\Gamma^{\lambda}_{\mu\nu} + g_{\lambda\mu}\Gamma^{\lambda}_{\kappa\nu}$$

Hint: start from the definition of the Christoffel symbol, and substitute into the RHS.

3. Employing the tensor transformation law for the metric tensor $g_{\mu\nu}$, show directly that the transformation law for the Christoffel symbol is

$$\Gamma'^{\kappa}_{\mu\nu} = \frac{\partial x'^{\kappa}}{\partial x^{\rho}} \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial x^{\beta}}{\partial x'^{\nu}} \Gamma^{\rho}_{\alpha\beta} + \frac{\partial x'^{\kappa}}{\partial x^{\rho}} \frac{\partial^2 x^{\rho}}{\partial x'^{\mu} \partial x'^{\nu}}$$

4. The surface of a sphere of radius a is characterised by the metric

$$d\ell^2 = a^2 d\theta^2 + a^2 \sin^2\theta \, d\phi^2$$

where θ and ϕ are spherical polar coordinates. Taking $x^1 = \theta$, $x^2 = \phi$, evaluate all Christoffel symbols in this coordinate system.