INSTITUT D'ASTROPHYSIQUE DE PARIS
Unité mixte de recherche 7095 CNRS - Université Pierre et Marie Curie

Celebrating 40 years of Milgromian dynamics

University of St Andrews

Dipolar Dark Matter
& +

Luc Blanchet

Gravitation et Cosmologie ($\mathcal{G}\mathbb{R}\varepsilon\mathbb{C}\mathcal{O}$)
Institut d'Astrophysique de Paris

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Plan of the talk

1. The MOND empirical formula
2. Modified gravity theory based on the Khronon
3. The dielectric analogy of MOND
4. Dipolar dark matter in general relativity
5. Test of MOND in the Solar System

THE MOND EMPIRICAL FORMULA

Challenges with Λ-CDM at galactic scales

[McGaugh & Sanders 2004; Famaey & McGaugh 2012]

The Λ-CDM paradigm is very successful in cosmological perturbations but faces important challenges when compared to observations at galactic scales

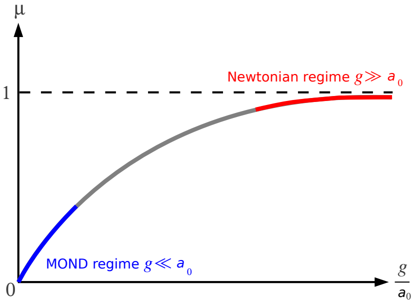
1. Unobserved predictions
   - Numerous but unseen satellites of large galaxies
   - Phase-space correlation of galaxy satellites
   - Generic formation of dark matter cusps in galaxies
   - Tidal dwarf galaxies dominated by dark matter
2. Unpredicted observations
   - Correlation between mass discrepancy and acceleration
   - Surface brightness of galaxies and the Freeman limit
   - Flat rotation curves of galaxies
   - Baryonic Tully-Fisher relation for spirals
   - Faber-Jackson relation for ellipticals

The MOND formula [Milgrom 1983abc; Bekenstein & Milgrom 1984]

The previous challenges are mysteriously solved by the MOND empirical formula

$$\boldsymbol{\nabla}\cdot\Bigg[\underbrace{\mu\left(\frac{g}{a_0}\right)}_{\text{MOND function}}\boldsymbol{g}\Bigg] = -4\pi\,G\,\rho_{\text{baryon}} \quad \text{with} \quad \boldsymbol{g} = \boldsymbol{\nabla}U$$

- The Newtonian regime is recoved when $g \gg a_0$
- In the MOND regime $g \ll a_0$ we have $\boxed{\mu \simeq g/a_0}$

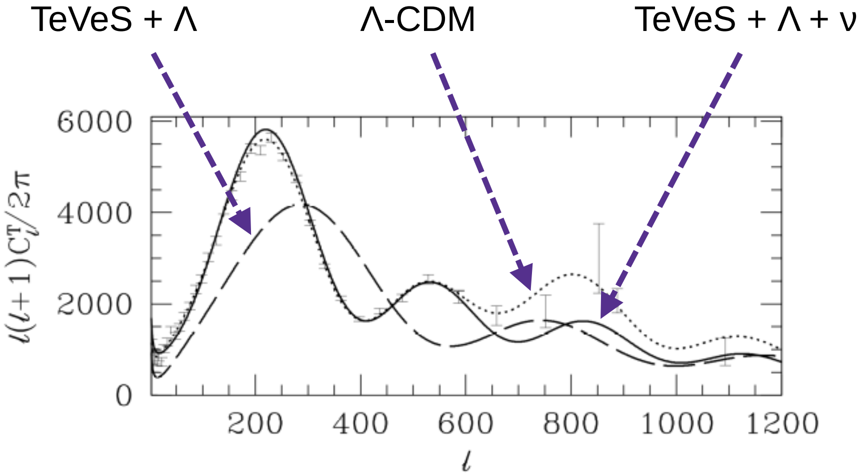


Different approaches to the dark matter problem

Faced with the "unreasonable effectiveness" of the MOND empirical formula three solutions are possible

1. Standard: MOND could be explained within the CDM paradigm
2. Modified Gravity: There is a fundamental modification of the law of gravity in a regime of weak gravity
   - Tensor-vector-scalar theory (TeVeS) [Bekenstein 2004; Sanders 2005]
   - Khronon theory [Blanchet & Marsat 2011; Sanders 2011]
   - Aether-scalar-tensor theory [Skordis & Zlosnik 2021; see the talk by Costantinos Skordis]
3. Modified Dark Matter: The law of gravity is not modified but DM is endowed with new properties able to explain the phenomenology of MOND

Importance of matching the standard cosmology



[Skordis, Mota, Ferreira & Boehm 2006]

MODIFIED GRAVITY THEORY BASED ON THE KHRONON

Non-canonical Einstein-Æther theories

1. These theories were introduced as a phenomenological approach to Lorentz-invariance violation [Jacobson & Mattingly 2001]

$$S_{\text{Æ}} = \frac{1}{16\pi}\int \mathrm{d}^4x\,\sqrt{-g}\Big[R + K[g,V] + \overbrace{\lambda\big(g^{\mu\nu}V_\mu V_\nu + 1\big)}^{\text{Lagrange multiplier constraint}}\Big]$$

2. Here $K[g,V]$ represents the most general Lagrangian density that is quadratic in the derivatives of the vector field V^μ

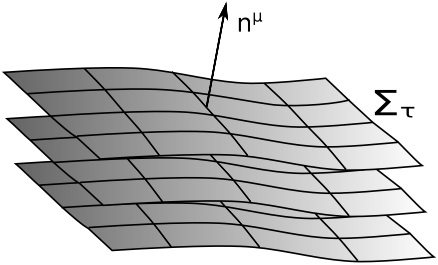
$$K = K^{\mu\nu\rho\sigma}\nabla_\mu V_\rho \nabla_\nu V_\sigma$$
$$K^{\mu\nu\rho\sigma} = c_1\, g^{\mu\nu}g^{\rho\sigma} + c_2\, g^{\mu\rho}g^{\nu\sigma} + c_3\, g^{\mu\sigma}g^{\nu\rho} + c_4\, V^\mu V^\nu g^{\rho\sigma}$$

3. Vector k-essence generalization of Einstein-Æther theories [Zlosnik, Ferreira & Starkman 2007; Halle, Zhao & Li 2008]

$$\boxed{S_{k\text{-essence Æ}} = \frac{1}{16\pi}\int \mathrm{d}^4x\,\sqrt{-g}\Big[R + F(K) + \lambda\big(g^{\mu\nu}V_\mu V_\nu + 1\big)\Big]}$$

where $F(K)$ is related *in fine* to the MOND function

From the Æther to the Khronon [Jacobson 2010; Blanchet & Marsat 2011]



The vector field is chosen to be hypersurface orthogonal

$$n_\mu = -N\partial_\mu \tau$$

where τ is a dynamical scalar field called the Khronon and where

$$N = \frac{1}{\sqrt{-g^{\rho\sigma}\partial_\rho\tau\partial_\sigma\tau}}$$

We have at our disposal the acceleration of the congruence of worldlines n_μ orthogonal to the foliation

$$a_\mu = n^\nu \nabla_\nu n_\mu = D_\mu \ln N$$

Since MOND is a modification of gravity in the weak-acceleration regime it is natural to build a theory using the acceleration vector a^μ

Simple relativistic MOND theory [Blanchet & Marsat 2011; Sanders 2011]

1. The dynamical degrees of freedom consist of the metric $g_{\mu\nu}$, the Khronon τ and the matter fields Ψ (essentially the baryons)
2. The covariant formulation of the theory reads

$$\boxed{S = \frac{1}{16\pi}\int \mathrm{d}^4x\,\sqrt{-g}\,\Big[R - 2f(a)\Big] + S_m\big[\Psi, g\big]}$$

where $f(a)$ is a function of the covariant acceleration squared $\boxed{a \equiv \sqrt{a_\mu a^\mu}}$

3. Varying with respect to the metric yields the modified Einstein field equation

$$G^{\mu\nu} + f(a)g^{\mu\nu} + 2n^\mu n^\nu \nabla_\rho\big[\chi(a)a^\rho\big] - 2\chi(a)a^\mu a^\nu = 8\pi\, T^{\mu\nu}$$

where we pose $\chi(a) = \frac{f'(a)}{2a}$.

4. Varying with respect to the Khronon gives $\nabla_\mu S^\mu = 0$ where

$$S^\mu = N\Big[n^\mu \nabla_\nu\left(\chi\, a^\nu\right) - \chi\, a^\mu n^\nu \nabla_\nu \ln N\Big]$$

Simple relativistic MOND theory [Blanchet & Marsat 2011; Sanders 2011]

1. The theory can be described in adapted coordinates where $t = \tau$ and the metric takes the standard 3+1 form

$$ds^2 = -(N^2 - N_i N^i)\mathrm{d}t^2 + 2N_i \mathrm{d}t\mathrm{d}x^i + \gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j$$

2. The theory involves then only N, N_i, γ_{ij} as dynamical variables ($a_\mu = D_\mu \ln N$) and we have

$$\boxed{S = \frac{1}{16\pi}\int \mathrm{d}^4x\,\sqrt{\gamma}N\Big[\mathcal{R} + K_{ij}K^{ij} - K^2 - 2f(a)\Big] + S_m[N, N_i, \gamma_{ij}, \Psi]}$$

3. The field equations are equivalent to the covariant equations and one of them plays the role of a modified Poisson equation

$$D_i\left[(1+\chi)a^i\right] + f + a^2 - \frac{1}{N}D_t K - K^{ij}K_{ij} = 4\pi\Big(\varepsilon + \frac{2}{N}N_i J^i + \mathcal{T}\Big)$$

4. In this coordinate system the theory exhibits a violation of Lorentz invariance

Non relativistic limit and MOND

1. Use the usual post-Newtonian ansatz for the metric

$$N = 1 + \frac{\phi}{c^2} + \mathcal{O}\left(\frac{1}{c^4}\right) ,$$

$$N_i = \mathcal{O}\left(\frac{1}{c^3}\right) ,$$

$$\gamma_{ij} = \delta_{ij}\left(1 - \frac{2\psi}{c^2}\right) + \mathcal{O}\left(\frac{1}{c^4}\right)$$

2. From the ij components of the Einstein field equation we obtain

$$\phi = \psi + \mathcal{O}\left(\frac{1}{c^2}\right)$$

which implies that the light deflection and gravitational lensing are given by the same formula as in GR

3. In the non-relativistic limit the acceleration reduces to the Newtonian one

$$a^i = \partial_i \phi + \mathcal{O}\left(\frac{1}{c^4}\right)$$

Non relativistic limit and MOND

1. The equation satisfied by the Newtonian potential ϕ is given by

$$\boxed{\partial_i\Big[(1+\chi)\,\partial_i\phi\Big] = 4\pi G\rho + \mathcal{O}\left(\frac{1}{c^2}\right)}$$

   which takes the MOND form with MOND function $\mu = 1 + \chi$

2. The Khronon equation imposes the constraint that the system should be stationary: $\dot{\phi} = 0$ [Flanagan 2023]

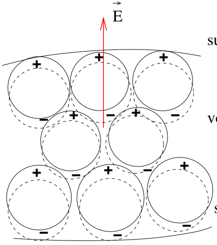
3. The theory reproduces the MOND phenomenology in the low acceleration regime for stationary systems and GR plus a cosmological constant for high accelerations with the function

$$\boxed{f(a) = \begin{cases} \Lambda & \text{when} \quad a \gg a_0 \\ -a^2 + \dfrac{2a^3}{3a_0} + \mathcal{O}\left(a^4\right) & \text{when} \quad a \ll a_0 \end{cases}}$$

4. However this theory does not reproduce the dark matter we see at cosmological scales [Blanchet & Skordis, in progress]

THE DIELECTRIC ANALOGY OF MOND

Electrostatics of dielectric media



1. In the presence of a polarization field the Gauss equation can be written as

$$\nabla \cdot \boldsymbol{E} = \frac{\rho_e + \rho_e^{\text{polar}}}{\varepsilon_0} \quad \Longleftrightarrow \quad \nabla \cdot \Big[\overbrace{(1+\chi_e)\boldsymbol{E}}^{\text{electric induction } \boldsymbol{D}} \Big] = \frac{\rho_e}{\varepsilon_0}$$

2. The density of polarization charges is

$$\rho_e^{\text{polar}} = -\nabla \cdot \boldsymbol{\Pi}_e$$

3. The polarization vector $\boldsymbol{\Pi}_e$ is aligned with the electric field

$$\boldsymbol{\Pi}_e = \varepsilon_0 \, \chi_e \, \boldsymbol{E}$$

where χ_e denotes the coefficient of electric susceptibility

Modified matter interpretation of MOND [Blanchet 2006]

1. The MOND equation in the form of a modified Poisson equation

$$\nabla \cdot \Big[\underbrace{\mu\left(\frac{g}{a_0}\right)}_{\text{MOND function}} \boldsymbol{g} \Big] = -4\pi \, G \, \rho_{\mathsf{b}}$$

is analogous to the equation of electrostatics inside a dielectric. We pose

$$\mu = 1 + \underbrace{\chi(g)}_{\substack{\text{gravitational}\\ \text{susceptibility}}} \qquad \text{and} \qquad \underbrace{\boldsymbol{\Pi}}_{\substack{\text{gravitational}\\ \text{polarization}}} = -\frac{\chi}{4\pi \, G} \boldsymbol{g}$$

2. The MOND equation is equivalent to

$$\boxed{\Delta U = -4\pi \, G \, \left(\rho_{\mathsf{b}} + \rho_{\text{polar}}\right)}$$

In this interpretation the Newtonian law of gravity is not violated but we are postulating a new form of DM consisting of polarization masses with density

$$\boxed{\rho_{\text{polar}} = -\nabla \cdot \boldsymbol{\Pi}}$$

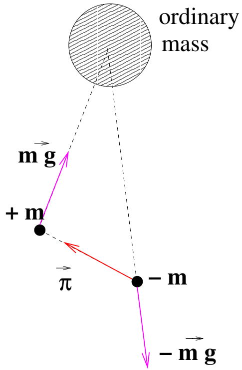
Microscopic description of the digravitational DM ?

Following the electrostatic analogy we suppose that

1. Dark matter consists of a digravitational medium made of individual gravitational dipole moments
$$\boldsymbol{\pi} = m\,\boldsymbol{\xi}$$
2. Each dipole moment is interpreted as a doublet of particles
$$(m_\mathrm{i}, m_\mathrm{g}) = (m, \pm m)$$
3. The polarization field is the density of dipole moments
$$\boldsymbol{\Pi} = m\,n\,\boldsymbol{\xi}$$
where n is the number density of the dipolar particles

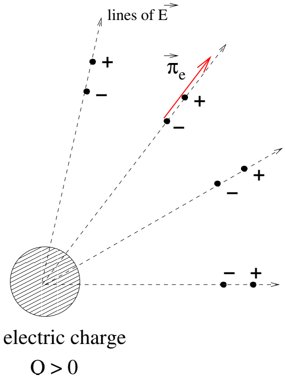
Therefore the model involves negative gravitational masses so violates the equivalence principle at a fundamental level

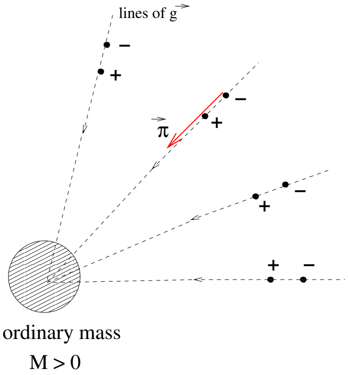
Microscopic description of the digravitational DM ?



- The dipole moments tend to align in the same direction as the gravitational field thus $\chi < 0$ which is exactly what MOND predicts
- Since the constituents of the dipole will repel each other we need to invoke a non-gravitational force (i.e. a fifth force) to stabilize the dipolar medium

Gravitational anti-screening





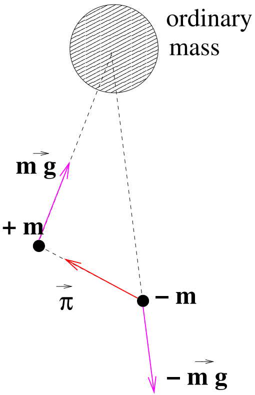
Screening by polarization charges

$\chi_e > 0$

Anti-screening by polarization masses

$\chi < 0$

Equations of motion of dipolar particles



$$m\,\frac{\mathrm{d}^2\mathbf{x}_+}{\mathrm{d}t^2} = m\,\mathbf{g}(\mathbf{x}_+) - \mathbf{F}(|\mathbf{x}_+ - \mathbf{x}_-|)$$

$$m\,\frac{\mathrm{d}^2\mathbf{x}_-}{\mathrm{d}t^2} = -m\,\mathbf{g}(\mathbf{x}_-) + \mathbf{F}(|\mathbf{x}_+ - \mathbf{x}_-|)$$

where $\mathbf{F}$ is an attractive non-gravitational force.

$$\mathbf{x} \equiv \frac{\mathbf{x}_+ + \mathbf{x}_-}{2} \qquad \boldsymbol{\xi} \equiv \mathbf{x}_+ - \mathbf{x}_- \qquad \boldsymbol{\pi} = m\,\boldsymbol{\xi}$$

$$\left.\begin{aligned} 2\,\frac{\mathrm{d}^2\mathbf{x}}{\mathrm{d}t^2} &= (\boldsymbol{\xi}\cdot\boldsymbol{\nabla})\,\mathbf{g} \\ m\,\frac{\mathrm{d}^2\boldsymbol{\xi}}{\mathrm{d}t^2} &= 2m\,\mathbf{g} - 2\mathbf{F} \end{aligned}\right\} + \mathcal{O}(\xi^2)$$

The dipolar particles are accelerated by the tidal gravitational field and therefore are weakly influenced by the distribution of ordinary matter

Interpretation of the dark matter medium

1. From the evolution equation of the dipole a situation of equilibrium is possible when the internal force compensates the gravitational force
$$\mathbf{F} = m\,\boldsymbol{g}$$
2. At equilibrium the dipole and polarization $\boldsymbol{\Pi} = n\,\boldsymbol{\pi}$ are aligned with the gravitational field. This provides a mechanism to verify the crucial relation
$$\boldsymbol{\Pi} = -\frac{\chi}{4\pi\,G}\,\boldsymbol{g}$$
3. Out of equilibrium we find that the dipole moment obeys the equation of an harmonic oscillator with plasma frequency
$$\frac{\mathrm{d}^2\boldsymbol{\xi}}{\mathrm{d}t^2} + \omega^2\,\boldsymbol{\xi} = 2\,\mathbf{g} \qquad \text{where} \qquad \omega = \sqrt{-\frac{8\pi\,G\,m\,n}{\chi}}$$
4. The DM medium is interpreted as the gravitational analogue of a plasma of particles with mass $(m_\mathrm{i}, m_\mathrm{g}) = (m, \pm m)$

DIPOLAR DARK MATTER IN GENERAL RELATIVITY

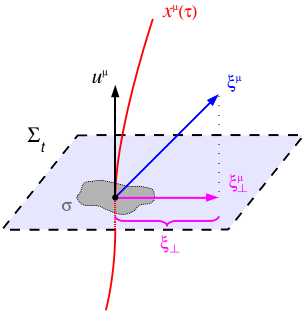
Dipolar fluid in general relativity [Blanchet & Le Tiec 2008; 2009]

The total action in standard GR is of the type

$$S = \int \mathrm{d}^4x \sqrt{-g} \left[\frac{c^3 R}{16\pi G} - \rho_\mathsf{b} \right] + S_\mathsf{DM}$$

Non-standard action for the dark matter

$$\boxed{S_\mathsf{DM} = \int \mathrm{d}^4x \sqrt{-g}\, L_\mathsf{DM} \left[J^\mu, \xi^\mu, \dot{\xi}^\mu, g_{\mu\nu} \right]}$$



The current density J^μ and the dipole moment ξ^μ are two independent dynamical variables

- The current density $J^\mu = \sigma u^\mu$ is conserved

$$\nabla_\mu J^\mu = 0$$

- The covariant time derivative is denoted

$$\dot{\xi}^\mu \equiv \frac{\mathrm{D}\xi^\mu}{\mathrm{d}\tau} = u^\nu \nabla_\nu \xi^\mu$$

Lagrangian for the dipolar fluid [Blanchet & Le Tiec 2008; 2009]

$$S_{\text{DM}} = \int \mathrm{d}^4x \sqrt{-g}\, \left[-\sigma + J^\mu \dot{\xi}_\mu - \mathcal{W}(\Pi_\perp)\right]$$

1. Mass term σ in an ordinary sense (like for ordinary CDM)
2. Interaction term between the fluid's mass current $J^\mu = \sigma u^\mu$ and the dipole moment
3. Potential term $\mathcal{W}$ describing an internal force and depending on the norm of the polarization $\Pi_\perp = \sigma\, \xi_\perp$

One easily proves that the only dynamical degrees of freedom of the dipole moment are the space-like projection orthogonal to the velocity

$$\xi_\perp^\mu = \perp^\mu_\nu \xi^\nu \quad \text{where the projector is} \quad \perp^\mu_\nu = \delta^\mu_\nu + u^\mu u_\nu$$

Hence the dipole vector moment is always space-like (in contract with modified gravity theory where the vector is time-like)

Equations of motion and evolution

Variation with respect to ξ^μ

Equation of motion of the dipolar fluid

$$\underbrace{\dot{u}^\mu = -\mathcal{F}^\mu}_{\text{non-geodesic motion}} \qquad \text{where} \qquad \underbrace{\mathcal{F}^\mu \equiv \hat{\xi}^\mu_\perp\, \mathcal{W}'}_{\text{dipolar internal force}}$$

Variation with respect to J^μ

Evolution equation of the dipole moment

$$\dot{\Omega}^\mu = \frac{1}{\sigma}\nabla^\mu\left(\mathcal{W} - \Pi_\perp \mathcal{W}'\right) - \underbrace{\xi^\nu_\perp\, R^\mu{}_{\rho\nu\sigma} u^\rho u^\sigma}_{\text{coupling to Riemann curvature}}$$

$$\text{where} \quad \Omega^\mu \equiv \dot{\xi}^\mu_\perp + u^\mu\left(1 + 2\xi_\perp \mathcal{W}'\right)$$

Stress-energy tensor

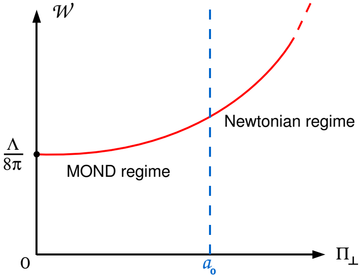
Variation with respect to $g_{\mu\nu}$

$$\begin{aligned} T^{\mu\nu} &= \Omega^{(\mu} J^{\nu)} && \Longleftarrow \text{monopolar DM} \\ &-\nabla_\rho \left(\left[\Pi_\perp^\rho u^{(\mu} - u^\rho \Pi_\perp^{(\mu} \right] u^{\nu)} \right) && \Longleftarrow \text{dipolar DM} \\ &-g^{\mu\nu} \left(\mathcal{W} - \Pi_\perp \mathcal{W}' \right) && \Longleftarrow \text{DE} \end{aligned}$$

The DM mass density is made of a monopolar term σ plus a dipolar term $-\nabla_\mu \Pi_\perp^\mu$ which appears as the relativistic analogue of the polarization mass density

$$\varepsilon \equiv u_\mu u_\nu T^{\mu\nu} = \underbrace{\sigma - \nabla_\mu \Pi_\perp^\mu}_{\text{DM energy density}} + \underbrace{\mathcal{W} - \Pi_\perp \mathcal{W}'}_{\text{DE}}$$

The internal potential



The potential $\mathcal{W}$ is phenomenologically determined through third order

$$\mathcal{W} = \frac{\Lambda}{8\pi} + 2\pi\,\Pi_\perp^2 + \frac{16\pi^2}{3a_0}\,\Pi_\perp^3 + \mathcal{O}\left(\Pi_\perp^4\right)$$

- The minimum of that potential is the cosmological constant Λ and the third-order deviation from the minimum contains the MOND scale a_0
- In this unification scheme the natural order of magnitude of the cosmological constant should be comparable with a_0

Cosmological perturbation at large scales

Consider a linear perturbation of the FLRW background. Since the dipole moment is space-like, it will break the spatial isotropy of the background, and must belong to the first-order perturbation

$$\xi_\perp^\mu = \mathcal{O}(1)$$

The stress-energy tensor reads $T^{\mu\nu} = T^{\mu\nu}_{\text{de}} + T^{\mu\nu}_{\text{dm}}$ where

- the DE is given by the cosmological constant Λ
- the DM takes the form of a perfect fluid with zero pressure

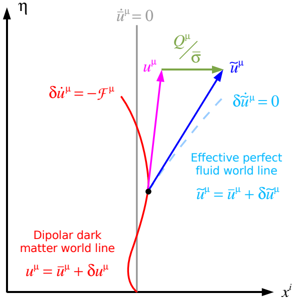
$$T^{\mu\nu}_{\text{dm}} = \rho\, \tilde{u}^\mu \tilde{u}^\nu + \mathcal{O}(2)$$

Here $\tilde{u}^\mu = u^\mu - \mathcal{L}_{\xi_\perp} u^\mu$ denotes an effective four-velocity field and

$$\rho \equiv \sigma - \nabla_\mu \Pi_\perp^\mu$$

is the energy density of the DM fluid

Agreement with the Λ-CDM scenario



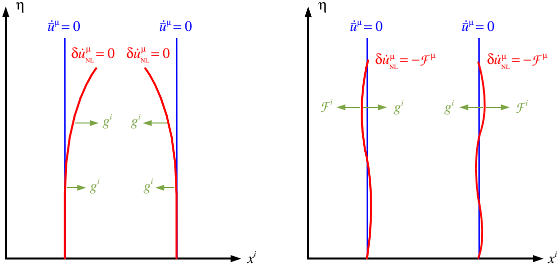
The dipolar fluid is undistinguishable from

- standard DE (a cosmological constant)
- standard CDM (a pressureless perfect fluid)

at the level of first-order cosmological perturbations

Adjusting Λ so that $\Omega_{\mathrm{de}} \simeq 0.73$ and $\overline{\sigma}$ so that $\Omega_{\mathrm{dm}} \simeq 0.23$ the model is consistent with CMB fluctuations

Weak clustering of dipolar DM



- Baryonic matter follows the geodesic equation $\dot{u}^\mu = 0$, therefore collapses in regions of overdensity
- Dipolar dark matter obeys $\dot{u}^\mu = -\mathcal{F}^\mu$, with the internal force $\boldsymbol{\mathcal{F}}$ balancing the gravitational field $\boldsymbol{g}$ created by an overdensity

The mass density of dipolar dark matter in a galaxy at low redshift should be smaller than the baryonic density and maybe close to its mean cosmological value

$$\sigma \approx \overline{\sigma} \ll \rho_\mathrm{b} \quad \text{and} \quad \boldsymbol{v} \approx \boldsymbol{0}$$

Non-relativistic limit of the model

$$\mathcal{L}_{\mathsf{DDM}} = \sigma\left(\frac{\boldsymbol{v}^2}{2} + U + \boldsymbol{g}\cdot\boldsymbol{\xi}_\perp + \boldsymbol{v}\cdot\frac{\mathrm{d}\boldsymbol{\xi}_\perp}{\mathrm{d}t}\right) - \mathcal{W}(\Pi_\perp) + \mathcal{O}\left(\frac{1}{c^2}\right)$$

We recognize the gravitational analogue $\boldsymbol{g}\cdot\boldsymbol{\Pi}_\perp$ of the coupling of the polarization field to an exterior field

- The equation of motion reads

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \boldsymbol{g} - \boldsymbol{\mathcal{F}}$$

- The gravitational equation is

$$\boldsymbol{\nabla}\cdot\left(\boldsymbol{g} - 4\pi\,\boldsymbol{\Pi}_\perp\right) = -4\pi\left(\rho_{\mathsf{b}} + \sigma\right)$$

Recovering MOND in a galaxy at low red-shift

Crucial use is made of the weak clustering of dipolar DM

- Using $\boldsymbol{v} \approx \mathbf{0}$ in the equation of motion

$$\boldsymbol{g} = \boldsymbol{\mathcal{F}} = \hat{\mathbf{\Pi}}_\perp \, \mathcal{W}' \implies \text{the dipolar medium is polarized}$$

- Using $\sigma \ll \rho_\mathsf{b}$ in the field equation

$$\nabla \cdot \left[\boldsymbol{g} - 4\pi \, \mathbf{\Pi}_\perp \right] = -4\pi \, \rho_\mathsf{b} \implies \text{the galaxy appears essentially baryonic}$$

Hence the MOND equation is recovered with MOND function $\mu = 1 + \chi$ such that

$$\boldsymbol{g} = \hat{\mathbf{\Pi}}_\perp \, \mathcal{W}' \iff \mathbf{\Pi}_\perp = -\frac{\chi(g)}{4\pi} \, \boldsymbol{g}$$

The model has recently been implemented numerically with the RAMSES code [Stahl, Montandon, Famaey, Hahn & Ibata 2022]

TEST OF MOND IN THE SOLAR SYSTEM

What about the Solar System scale?

1. In spherical symmetry the MOND equation becomes

$$\mu\left(\frac{g}{a_0}\right) g = g_{\mathsf{N}} \equiv \frac{GM_\odot}{r^2}$$

2. Suppose MOND approaches the Newtonian regime like

$$\mu\left(\frac{g}{a_0}\right) = 1 - k\left(\frac{a_0}{g}\right)^q \quad \text{when} \quad g \to \infty$$

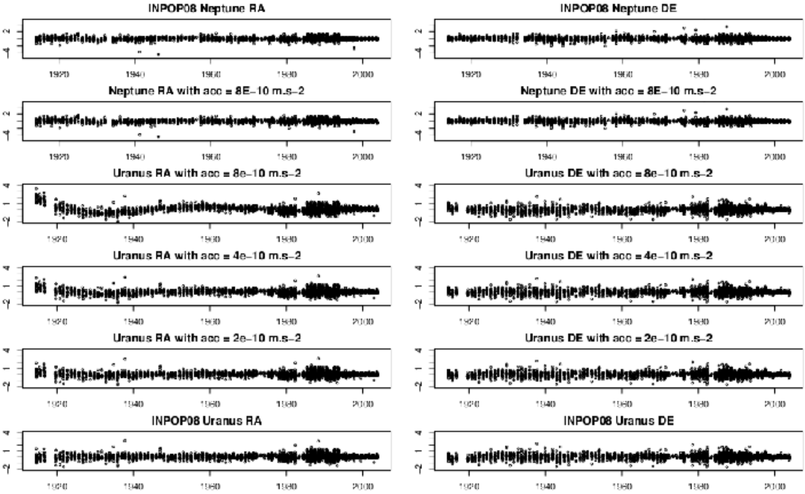
3. With $r_0 = \sqrt{GM_\odot / a_0}$ the MOND transition radius for the Sun

$$g = g_{\mathsf{N}} + k\, a_0 \left(\frac{r}{r_0}\right)^{2q-2}$$

When $q = 1$ this gives a Pioneer-like anomaly

$$a_{\mathsf{P}} = k\, a_0$$

Solar System data and Pioneer anomaly [Fienga *et al.* 2009]



The data exclude a Pioneer-like anomaly at the level $5 \times 10^{-13} \mathrm{m/s}^2$

The external field effect in MOND [Milgrom 1983]

1. Open star clusters in our Galaxy do not show evidence for dark matter despite their typical low internal gravity $g_\text{i} \ll a_0$
2. In the presence of the external Galactic field g_e the MOND equation which is non-linear can be approximated by

$$\mu\left(\frac{|\boldsymbol{g}_\text{i}+\boldsymbol{g}_\text{e}|}{a_0}\right)\boldsymbol{g}_\text{i} \approx \boldsymbol{g}_\text{i}^\text{Newtonian}$$

- When $a_0 \lesssim g_\text{e}$ the sub-system exhibits Newtonian behaviour
- When $g_\text{i} \lesssim g_\text{e} \lesssim a_0$ the system is still Newtonian but with an effective Newton's constant G/μ_e

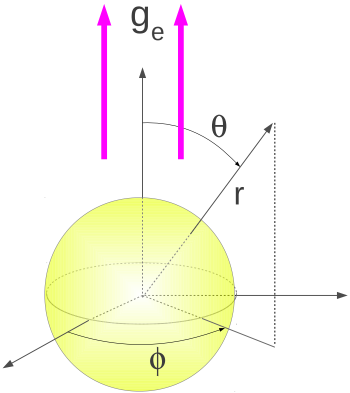
The EFE results from a violation of the strong version of the equivalence principle

The gravitational dynamics of a system is influenced by the external gravitational field in which the system is embedded

Deformation of the Sun's field by the Galactic field

The external field effect is a prediction of the non-linear Poisson equation

$$\nabla \cdot \left[\mu\left(\frac{g}{a_0}\right) \nabla U \right] = -4\pi\, G\, \rho_\mathsf{b}$$



The MOND field of the Sun, in the presence of the external field of the Galaxy, is deformed along the direction of the Galactic center

$$U = \boldsymbol{g}_\mathsf{e} \cdot \mathbf{x} + \frac{G M_\odot / \mu_\mathsf{e}}{r\sqrt{1 + \lambda_\mathsf{e} \sin^2\theta}} + \mathcal{O}\left(\frac{1}{r^2}\right)$$

This effect influences the motion of inner planets of the Solar System

Multipole expansion of the MOND field of the Sun

1. The Newtonian physicist measures from the motion of planets the internal gravitational potential $u = U - \boldsymbol{g}_{\mathsf{e}} \cdot \boldsymbol{x}$ and detects the anomaly

$$\delta u = u - u_{\mathsf{N}} = G \int \frac{\mathrm{d}^3\mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|}\, \rho_{\mathsf{pdm}}(\mathbf{x}', t)$$

2. Since the phantom dark matter vanishes in the strong-field regime near the Sun δu is an harmonic function and admits the multipole expansion

$$\delta u = \sum_{l=0}^{+\infty} \frac{(-)^l}{l!}\, x^L Q_L$$

where Q_L are trace-free multipolar coefficients

3. This expansion is valid in the region inside the MOND transition radius

$$r_0 = \sqrt{\frac{GM_\odot}{a_0}} \approx 7100\,\mathrm{AU}$$

Effect in the Solar System [Milgrom 2009, Blanchet & Novak 2011]

- The effect is dominantly quadrupolar and grows with the distance squared

$$u = \frac{GM_\odot}{r} + \frac{1}{2} x^i x^j Q_{ij}$$

- The quadrupole moment is aligned in the direction of the Galactic center

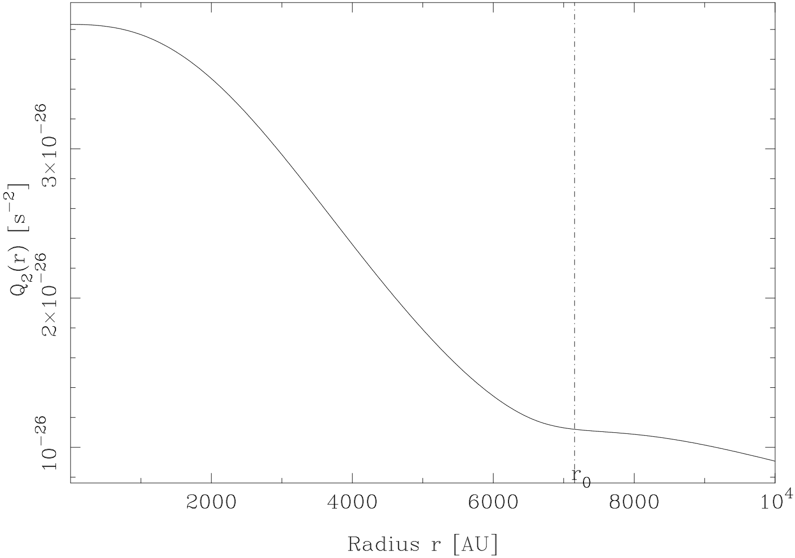
$$Q_{ij} = Q_2 \left(e_i e_j - \frac{1}{3}\delta_{ij} \right)$$

- The quadrupole moment is computed by solving numerically the MOND equation in the presence of the external galactic field. We find

$$2.1 \times 10^{-27}\ \mathrm{s}^{-2} \ \lesssim \ Q_2 \ \lesssim \ 4.1 \times 10^{-26}\ \mathrm{s}^{-2}$$

depending on the MOND function in use

Quadrupole moment as a function of distance



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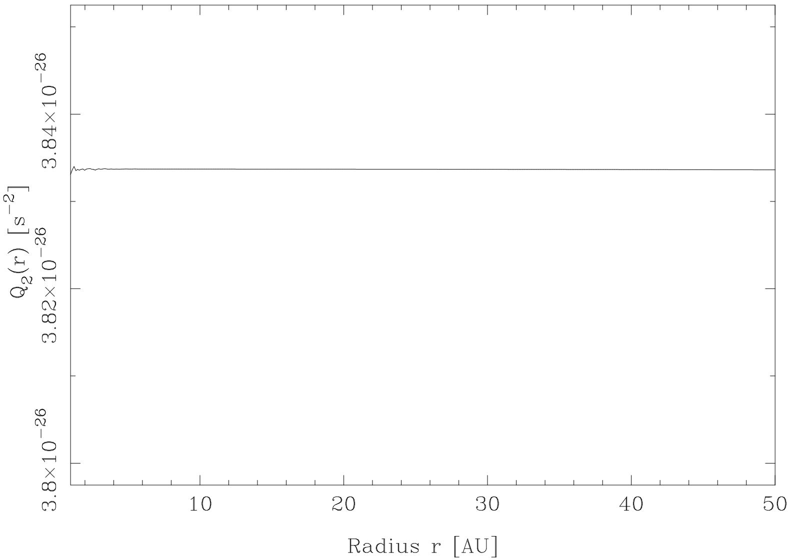

Effect on the dynamics of Solar System planets

The quadrupole effect yields a suplementary precession of the semi-major axis of planets of the Solar System [Blanchet & Novak 2011]

$$\left\langle \frac{\mathrm{d}e}{\mathrm{d}t} \right\rangle = \frac{5Q_2 e\sqrt{1-e^2}}{4n}\sin(2\tilde{\omega})$$

$$\left\langle \frac{\mathrm{d}\ell}{\mathrm{d}t} \right\rangle = n - \frac{Q_2}{12n}\left[7+3e^2+15(1+e^2)\cos(2\tilde{\omega})\right]$$

$$\left\langle \frac{\mathrm{d}\tilde{\omega}}{\mathrm{d}t} \right\rangle = \frac{Q_2\sqrt{1-e^2}}{4n}\left[1+5\cos(2\tilde{\omega})\right]$$

Comparison with Solar System ephemerides

Predicted values for the orbital precession

Quadrupolar precession rate in mas/cy						
	Mercury	Venus	Earth	Mars	Jupiter	Saturn
μ_1	0.04	0.02	0.16	-0.16	-1.12	5.39
μ_2	0.02	0.01	0.09	-0.09	-0.65	3.12
μ_5	7×10^{-3}	3×10^{-3}	0.03	-0.03	-0.22	1.05
μ_{20}	2×10^{-3}	10^{-3}	9×10^{-3}	-9×10^{-3}	-0.06	0.3

Best published residuals for orbital precession

Postfit residuals for the precession rates in mas/cy						
	Mercury	Venus	Earth	Mars	Jupiter	Saturn
[Pitjeva 2005]	-3.6 ± 5	-0.4 ± 0.5	-0.2 ± 0.4	0.1 ± 0.5	-	-6 ± 2
[Fienga *et al.* 2009]	-10 ± 30	-4 ± 6	0 ± 0.016	0 ± 0.2	142 ± 156	-10 ± 8
[Fienga *et al.* 2010]	0.4 ± 0.6	0.2 ± 1.5	-0.2 ± 0.9	0 ± 0.1	-41 ± 42	0.2 ± 0.7

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MOND (in the form proposed by [Bekenstein & Milgrom, 1984]) seems to be marginally excluded by planetary ephemerides