Lecture 6

General Relativity:
Field Equations

Dynamics of the Universe:
R(t) = ?
H(x) = ?

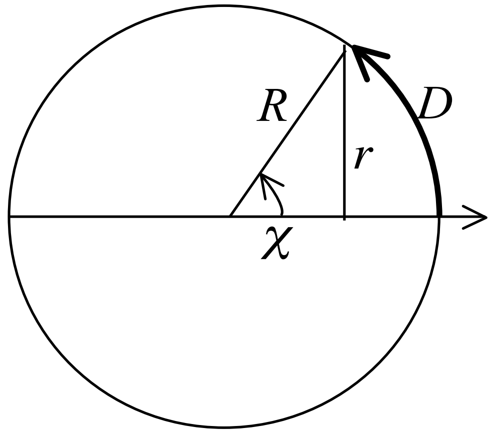
Friedmann Equation

Time and Distance vs Redshift

$$\frac{d}{dt}\left(x = 1 + z = \frac{R_0}{R} \right) \rightarrow \quad dt = \frac{-dx}{x\,H(x)}$$

Look - back time :

$$t(z) = \int_t^{t_0} dt = \int_{1+z}^{1} \frac{-dx}{x\,H(x)} = \int_1^{1+z} \frac{dx}{x\,H(x)}$$



Age : $\quad t_0 = t(z \rightarrow \infty)$

Distance : $\quad D = R\,\chi \qquad r = R\,S_k(\chi)$

$$\chi(z) = \int d\chi = \int_t^{t_0} \frac{c\,dt}{R(t)} = \frac{c}{R_0}\int_1^{1+z} \frac{R_0}{R(t)}\frac{dx}{x\,H(x)} = \frac{c}{R_0}\int_1^{1+z} \frac{dx}{H(x)}$$

Horizon : $\quad \chi_H = \chi(z \rightarrow \infty)$

Need to know *R(t)*, or *R_0* and *H(x)*.

Einstein's General Relativity

- 1. ***Spacetime geometry tells matter how to move***
  - gravity = effect of curved spacetime
  - free particles follow geodesic trajectories

- 2. ***Matter (energy) tells spacetime how to curve***
  - Einstein field equations
    - nonlinear
    - second-order derivatives of metric
      with respect to space/time coordinates

Einstein Field Equations

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R^{\alpha}{}_{\alpha}\, g_{\mu\nu} = \frac{8\pi G}{c^2} T_{\mu\nu} - \Lambda\, g_{\mu\nu}$$

$g_{\mu\nu}$ = spacetime metric ($ds^2 = g_{\mu\nu}\, dx^{\mu}\, dx^{\nu}$)

$G_{\mu\nu}$ = Einstein tensor (spacetime curvature)

$R_{\mu\nu}$ = Ricci curvature tensor

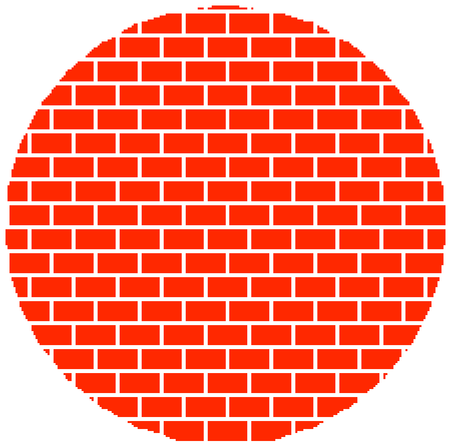
$R^{\alpha}{}_{\alpha}$ = Ricci curvature scalar

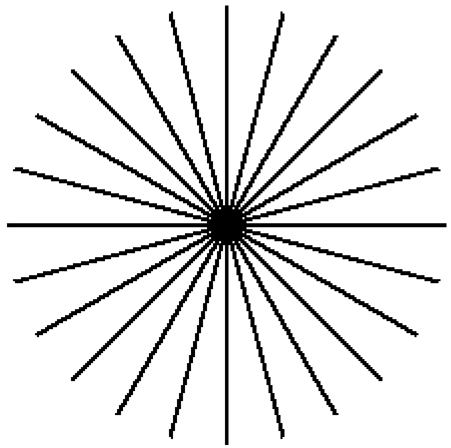
G = Netwon's gravitational constant

$T_{\mu\nu}$ = energy - momentum tensor

Λ = cosmological constant

Homogeneity and Isotropy





homogeneous

not isotropic

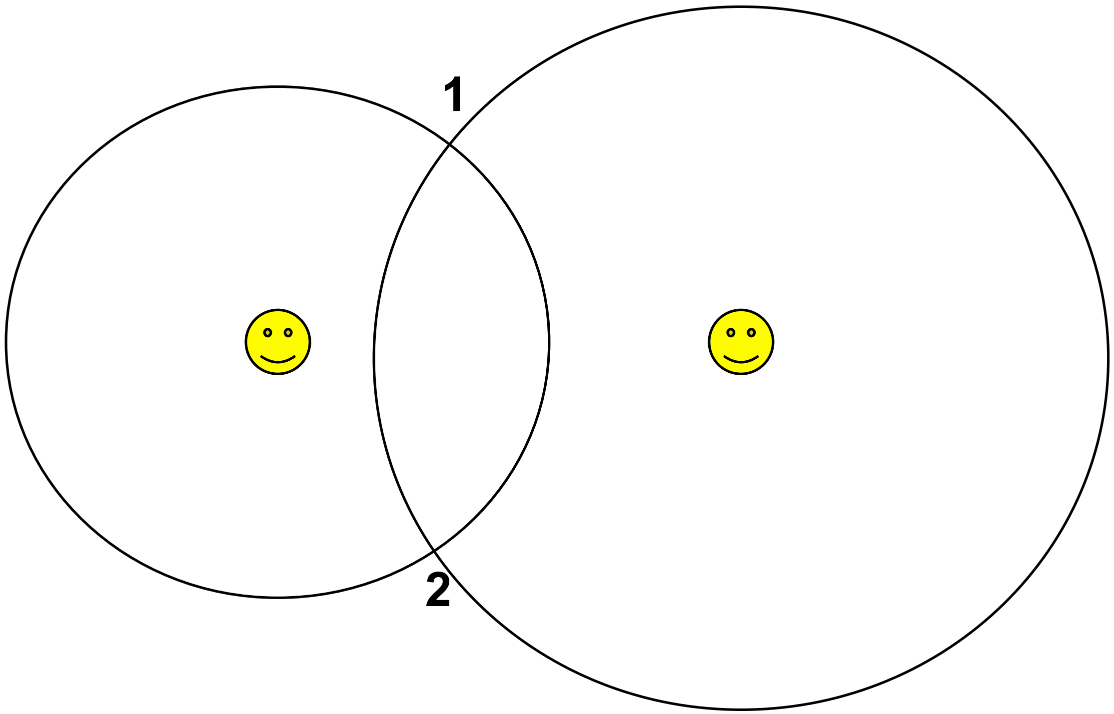
isotropic

not homogeneous

For cosmology, assume Universe is Homogeneous.

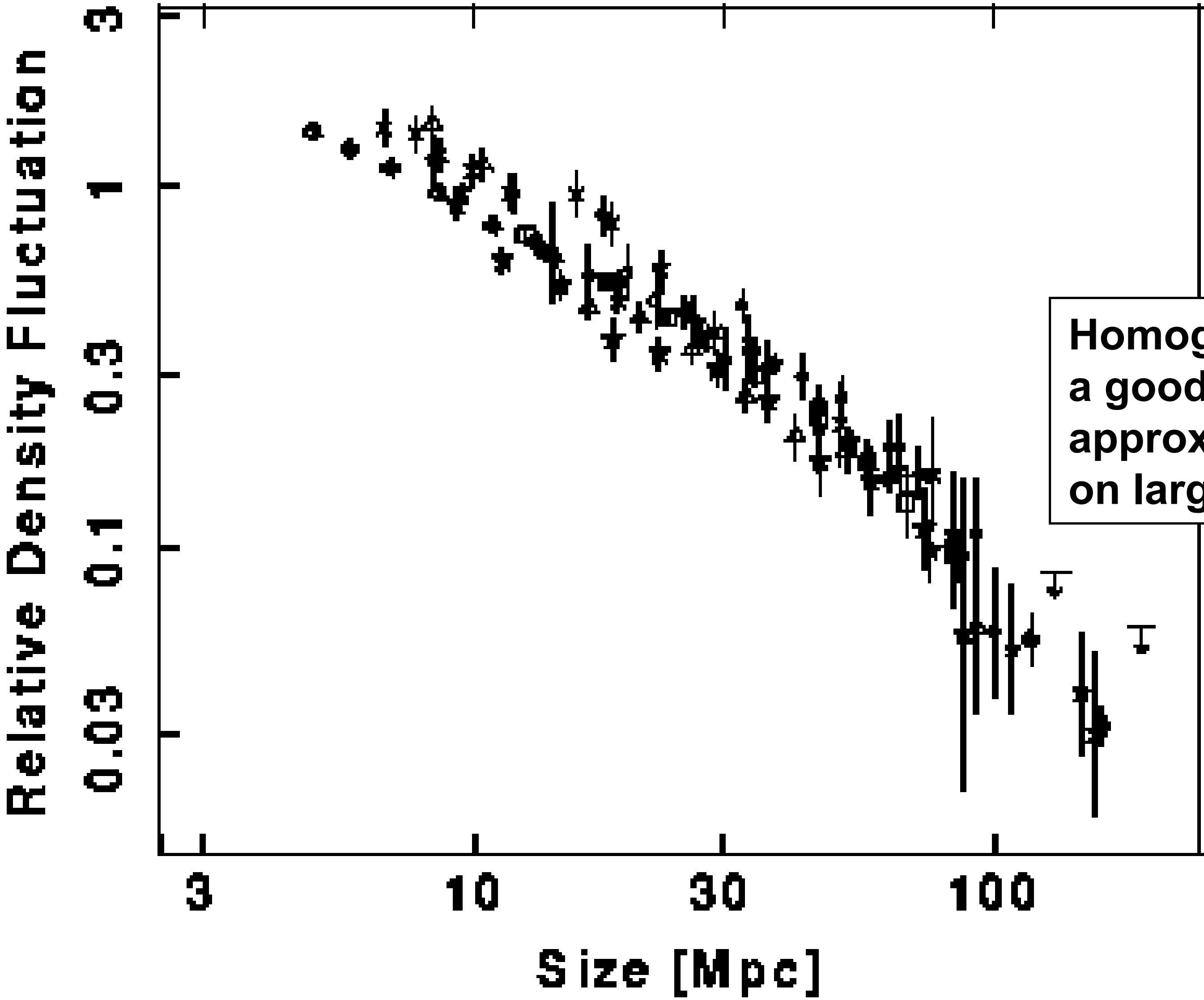
Simplifies the equations. :)

Cosmological Principle (assumed) + Isotropy (observed) => Homogeneity



$\rho_1 = \rho_2$ otherise not isotropic for equidistant fidos

Density Fluctuations vs Scale



Homogeneity is a good approximation on large scales.

Homogeneous perfect fluid

density ρ pressure p

Einstein field equations:

$$G_{\mu\nu} = \frac{8\pi G}{c^2}\begin{pmatrix} \rho c^2 & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} - \Lambda \begin{pmatrix} -c^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

---> Friedmann equations :

$$\dot{R}^2 = \left(\frac{8\pi G \rho + \Lambda}{3}\right) R^2 - k c^2$$

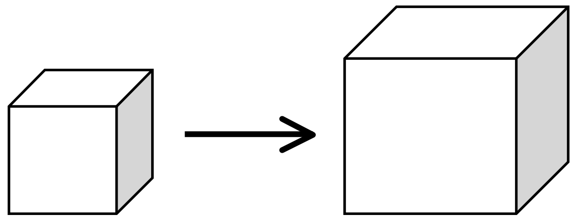
energy

$$\ddot{R} = -\frac{4\pi G}{3c^2}\left(\rho c^2 + 3p\right) R + \frac{\Lambda}{3} R$$

momentum

Note: energy density and pressure decelerate, Λ accelerates.

Local Conservation of Energy



$$d\left[energy\right] = work$$

$$d\left[\rho\, c^2 R^3\right] = -p\, d\left[R^3\right]$$

$$\dot{\rho}\, c^2 R^3 + \rho\, c^2\, (3\, R^2 \dot{R}) = -p\, (3\, R^2 \dot{R})$$

$$\dot{\rho} = -3\left(\rho + \frac{p}{c^2}\right)\frac{\dot{R}}{R} \qquad p = p(\rho) = \text{equation of state}$$

$$\text{Friedmann 1}: \dot{R}^2 = \frac{8\pi\, G}{3}\rho\, R^2 + \frac{\Lambda}{3} R^2 - k\, c^2$$

$$\left(2\, \dot{R}\, \ddot{R}\right) = \frac{8\pi\, G}{3}\left(\dot{\rho}\, R^2 + 2\, R\, \dot{R}\, \rho\right) + \frac{\Lambda}{3}\left(2\, R\, \dot{R}\right)$$

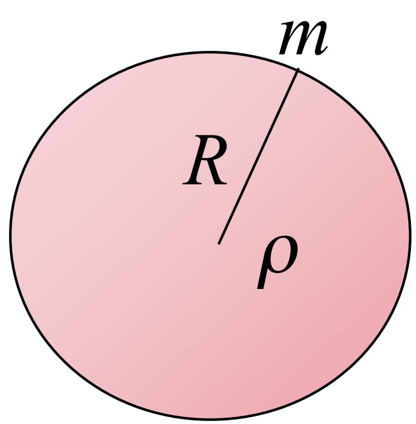
$$\ddot{R} = \frac{8\pi\, G}{3}\left(\frac{\dot{\rho}\, R^2}{2\, \dot{R}} + R\, \rho\right) + \frac{\Lambda}{3} R$$

$$\ddot{R} = -\frac{4\pi\, G}{3}\left(\rho + \frac{3\, p}{c^2}\right) R + \frac{\Lambda}{3} R = \text{Friedmann 2}$$

Newtonian Analogy

$$E = \frac{m}{2}\dot{R}^2 - \frac{G\ M\ m}{R} \qquad M = \frac{4\pi}{3}R^3\rho$$

$$\dot{R}^2 = \frac{8\pi\ G}{3}\rho\ R^2 + \frac{2E}{m}$$



Friedmann equation:

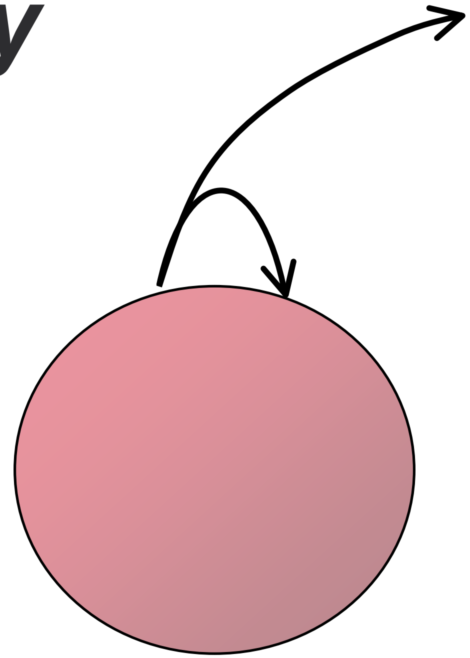
$$\dot{R}^2 = \left(\frac{8\pi\ G\ \rho + \Lambda}{3}\right)R^2 - k\ c^2$$

same equation if $\rho \rightarrow \rho + \frac{\Lambda}{8\pi\ \mathrm{G}}, \quad \frac{2E}{m} \rightarrow -k\ c^2$

Newtonian Analogy

$$E = \frac{m}{2}\dot{R}^2 - \frac{G\,M\,m}{R}$$

$$V_{esc} = \sqrt{\frac{2\,G\,M}{R}}$$



$$E > 0 \quad V > V_{esc} \quad R \to \infty \quad V_\infty > 0$$

$$E = 0 \quad V = V_{esc} \quad R \to \infty \quad V_\infty = 0$$

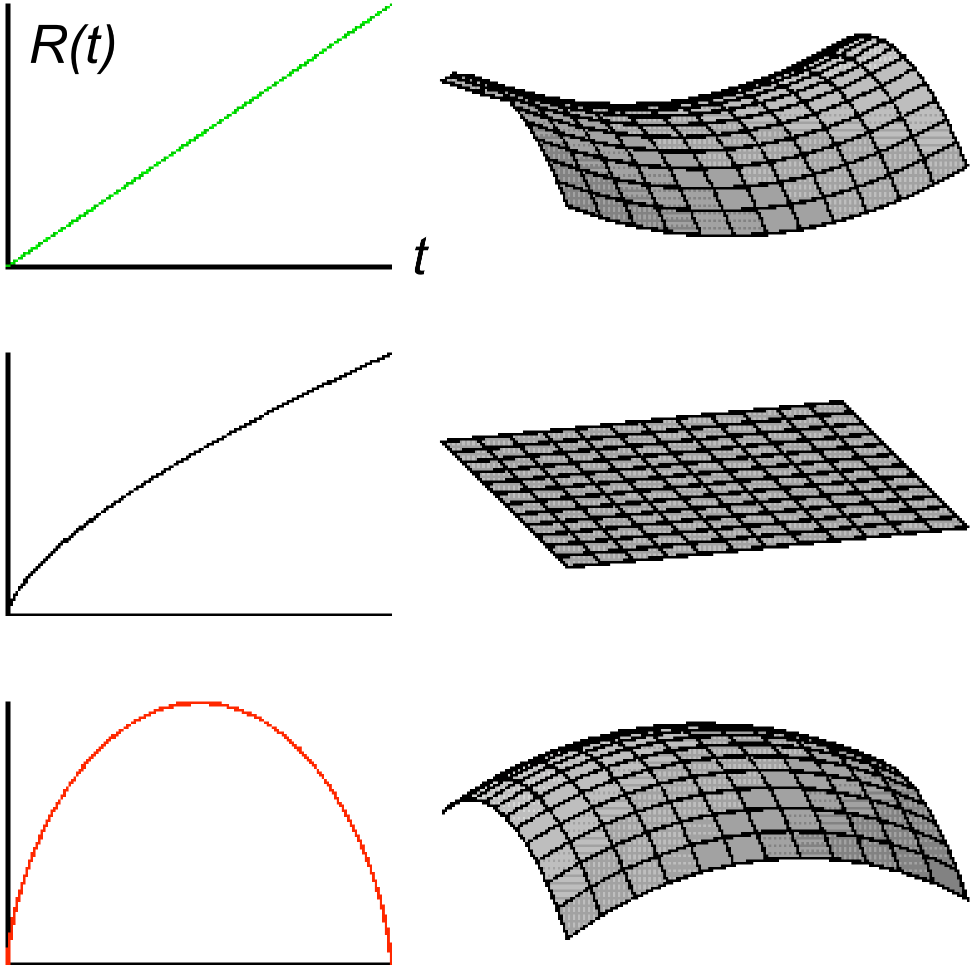
$$E < 0 \quad V < V_{esc} \quad R \to 0$$

Density - Evolution - Geometry

$\rho < \rho_c$

$\rho = \rho_c$

$\rho > \rho_c$



Open
$k = -1$

Flat
$k = 0$

Closed
$k = +1$

Critical Density

- **Derive using Newtonian analogy:**

escape velocity :

$$V_{esc}^{\ 2} = \frac{2\,G\,M}{R} = \frac{2\,G}{R}\left(\frac{4\pi\,R^3\,\rho}{3}\right) = \frac{8\pi\,G\,R^2\,\rho}{3}$$

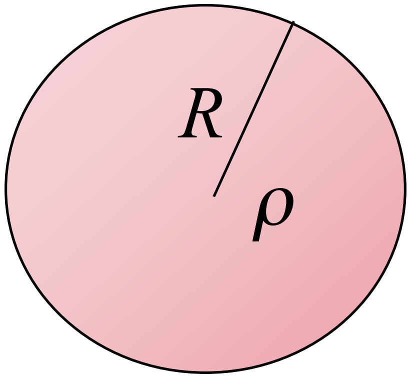
Hubble expansion :

$$V = \dot{R} = H_0\,R$$

critical density :

$$\left(\frac{V_{esc}}{V}\right)^2 = \frac{8\pi\,G\,\rho}{3\,H_0^{\ 2}} \equiv \frac{\rho}{\rho_c}$$

$$\rho_c = \frac{3\,H_0^{\ 2}}{8\pi\,G}$$



radiation -> matter -> vacuum

radiation: $\rho_R \propto R^{-4}$

matter: $\rho_M \propto R^{-3}$

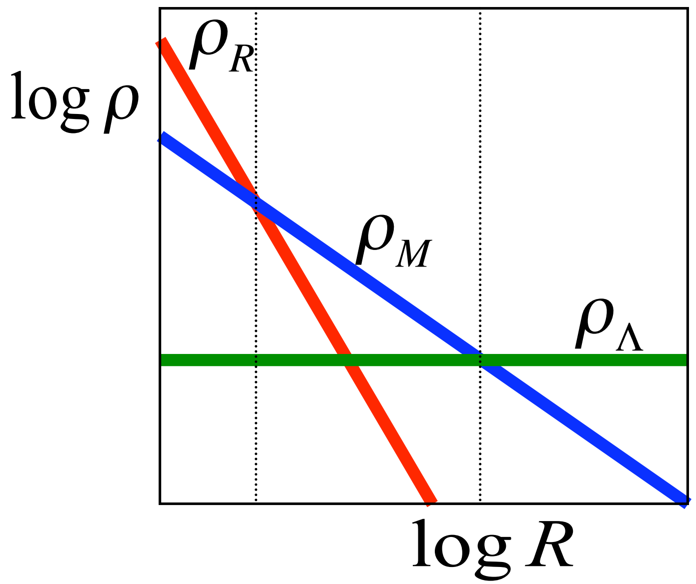
vacuum: $\rho_\Lambda = \text{const}$

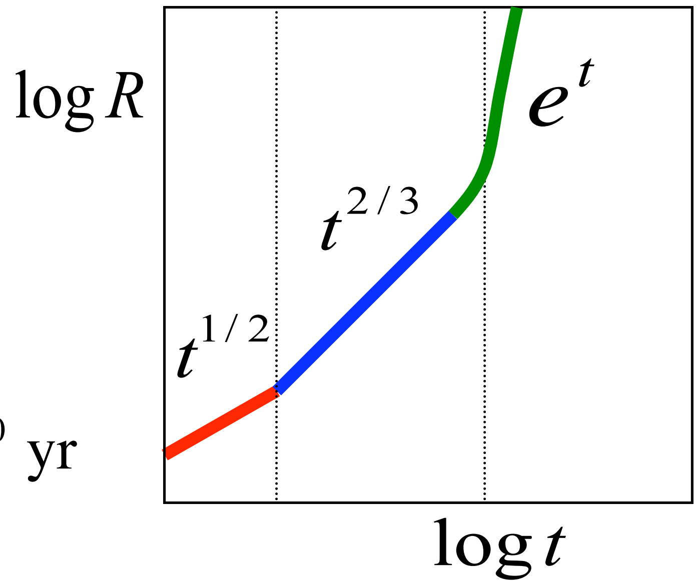
$$\rho(x) = \rho_R\, x^4 + \rho_M\, x^3 + \rho_\Lambda$$

$$x = 1 + z = \frac{R_0}{R} = \frac{1}{a}$$

$\rho_R = \rho_M$ at $x \sim \frac{\rho_M}{\rho_R} \sim 10^4 \quad t \sim 10^4$ yr

$\rho_M = \rho_\Lambda$ at $x \sim \left(\frac{\rho_\Lambda}{\rho_M}\right)^{1/3} \; z \sim 0.3 \quad t \sim 10^{10}$ yr



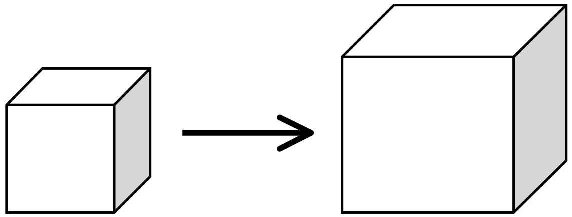


Equation of State ----- w

Equation of state :

$$\rho \propto R^{-n} \qquad n = 3(1+w)$$

$$w \equiv \frac{\text{pressure}}{\text{energy density}} = \frac{p}{\rho\, c^2} = \frac{n}{3} - 1$$



<u>Radiation</u> : $(n = 4,\, w = 1/3)$

$$p_R = \frac{1}{3} \rho_R\, c^2$$

<u>Matter</u> : $(n = 3,\, w = 0)$

$$p_M \sim \rho_M\, c_S^{\,2} << \rho_M\, c^2$$

<u>Vacuum</u> : $(n = 0,\, w = -1)$

$$p_\Lambda = -\rho_\Lambda\, c^2$$

Negative Pressure ! ?

$$d[energy] = work$$

$$d\left[\rho\, c^2 R^3\right] = -p\, d\left[R^3\right]$$

$$\rho\, c^2 \left(3\, R^2 dR \right) + R^3\, c^2 d\rho = -p \left(3\, R^2 dR \right)$$

$$1 + \frac{R\, d\rho}{3\, \rho\, dR} = -\frac{p}{\rho c^2} \equiv -w$$

$$w = -\frac{1}{3} \frac{d[\ln \rho]}{d[\ln R]} - 1$$

$$w = \frac{n}{3} - 1$$

Density Parameters

critical density:

$$\rho_c \equiv \frac{3\,H_0^{\;2}}{8\pi\,G}$$

density parameters (today):

$$\Omega_R \equiv \frac{\rho_R}{\rho_c} \qquad \Omega_M \equiv \frac{\rho_M}{\rho_c} \qquad \Omega_\Lambda \equiv \frac{\rho_\Lambda}{\rho_c} = \frac{\Lambda}{3\,H_0^{\;2}}$$

total density parameter today:

$$\Omega_0 \equiv \Omega_R + \Omega_M + \Omega_\Lambda$$

density at a past/future epoch in units of today's critical density:

$$\Omega \equiv \frac{\rho}{\rho_c} = \sum_w \Omega_w x^{3(1+w)} = \Omega_R x^4 + \Omega_M x^3 + \Omega_\Lambda \qquad x \equiv 1 + z = R_0 / R$$

in units of critical density at the past/future epoch:

$$\Omega(x) \equiv \frac{8\pi\,G\rho}{3H^2} = \frac{H_0^{\;2}}{H^2}\sum_w \Omega_w x^{3(1+w)} = \frac{\Omega_R x^4 + \Omega_M x^3 + \Omega_\Lambda}{\Omega_R x^4 + \Omega_M x^3 + \Omega_\Lambda + (1-\Omega_0)x^2}$$

Note: radiation dominates at high z, can be neglected at lower z.

Hubble Parameter Evolution -- H(z)

$$H^2 \equiv \left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3} - \frac{k c^2}{R^2}$$

$$\frac{H^2}{H_0^{\,2}} = \Omega_R x^4 + \Omega_M x^3 + \Omega_\Lambda - \frac{k c^2}{H_0^{\,2} R_0^{\,2}} x^2$$

$$\text{evaluate at } x = 1 \quad \rightarrow \quad 1 = \Omega_0 - \frac{k c^2}{H_0^{\,2} R_0^{\,2}}$$

Dimensionless Friedmann Equation:

$$\frac{H^2}{H_0^{\,2}} = \Omega_R x^4 + \Omega_M x^3 + \Omega_\Lambda + (1 - \Omega_0)\, x^2$$

$$x = 1 + z = R_0 / R$$

$$\rho_c = \frac{3 H_0^{\,2}}{8\pi G}$$

$$\Omega_M \equiv \frac{\rho_M}{\rho_c}, \quad \Omega_R \equiv \frac{\rho_R}{\rho_c}$$

$$\Omega_\Lambda \equiv \frac{\rho_\Lambda}{\rho_c} = \frac{\Lambda}{3 H_0^{\,2}}$$

$$\Omega_0 \equiv \Omega_M + \Omega_R + \Omega_\Lambda$$

Curvature Radius today:

$$R_0 = \frac{c}{H_0}\sqrt{\frac{k}{\Omega_0 - 1}} \rightarrow \begin{cases} k = +1 & \Omega_0 > 1 \\ k = 0 & \Omega_0 = 1 \\ k = -1 & \Omega_0 < 1 \end{cases}$$

Density determines Geometry

Possible Universes

$$H_0 \approx 70 \frac{\text{km/s}}{\text{Mpc}}$$

$$\Omega_M \sim 0.3$$

$$\Omega_\Lambda \sim 0.7$$

$$\Omega_R \sim 8\times10^{-5}$$

$$\Omega = 1.0$$