

Lecture 4

Space-Time Metric

AS 4022 Cosmology

Olber's Paradox

Why is the sky dark at night ?

Flux from all stars in the sky :

$$F = \int n_* F_* d(\text{Vol}) = \int_0^{\chi_{\max}} n_* \left(\frac{L_*}{A(\chi)} \right) (A(\chi) R d\chi)$$

$$= n_* L_* R \chi_{\max}$$

$$\Rightarrow \infty \text{ for flat space, } R \rightarrow \infty.$$

A dark sky may imply :

- (1) an edge (we don't observe one)
- (2) a curved space (finite size)
- (3) expansion ($R(t) \Rightarrow$ finite age, redshift)

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Minkowski Spacetime Metric

$$ds^2 = -c^2 dt^2 + dl^2$$

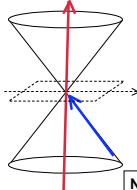
$$d\tau^2 = dt^2 - \frac{dl^2}{c^2} = dt^2 \left(1 - \frac{1}{c^2} \left(\frac{dl}{dt} \right)^2 \right)$$

Time-like intervals:
 $ds^2 < 0, d\tau^2 > 0$
 Inside light cone.
 Causally connected.

Proper time (moving clock):

$$d\tau = \sqrt{-ds^2}/c^2$$

$$= dt \sqrt{1 - \frac{v^2}{c^2}} > 0$$



Space-like intervals:
 $ds^2 > 0, d\tau^2 < 0$
 Outside light cone.
 Causally disconnected.

World line of massive particle at rest.

Null intervals light cone:
 $v = c, ds^2 = 0$

Photons arrive from our past light cone.

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Minkowski Spacetime Metric

$$ds^2 = -c^2 dt^2 + dl^2$$

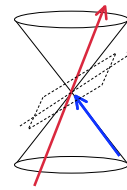
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 Inside light cone.
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Proper time (moving clock):

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$$= dt \sqrt{1 - \frac{v^2}{c^2}} > 0$$



Space-like intervals:
 $ds^2 > 0, d\tau^2 < 0$
 Outside light cone.
 Causally disconnected.

World line of massive particle in motion.

Null intervals light cone:
 $v = c, ds^2 = 0$

Photons arrive from our past light cone.

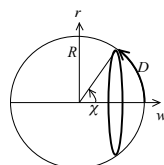
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Robertson-Walker metric uniformly curved, evolving spacetime

$$ds^2 = -c^2 dt^2 + R^2(t) (d\chi^2 + S_k^2(\chi) d\psi^2)$$

$$= -c^2 dt^2 + R^2(t) \left(\frac{du^2}{1-k u^2} + u^2 d\psi^2 \right)$$

$$= -c^2 dt^2 + a^2(t) \left(\frac{dr^2}{1-k(r/R_0)^2} + r^2 d\psi^2 \right)$$



$$S_k(\chi) = \begin{cases} \sin \chi & (k=+1) \text{ closed} \\ \chi & (k=0) \text{ flat} \\ \sinh \chi & (k=-1) \text{ open} \end{cases} \quad \begin{aligned} d\psi^2 &= d\theta^2 + \sin^2 \theta d\phi^2 \\ a(t) &= R(t)/R_0 \\ R_0 &= R(t_0) \end{aligned}$$

radial distance = $D(t) = R(t)\chi$

circumference = $2\pi r(t) \quad r(t) = a(t)r = R(t)u = R(t)S_k(\chi)$

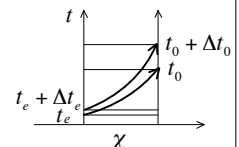
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Redshift and Time Dilation

Light rays are null geodesics:

$$ds^2 = R^2(t) d\chi^2 - c^2 dt^2 = 0$$

$$d\chi = \frac{c dt}{R(t)}$$



$$\chi = \int_{t_e}^{t_e + \Delta t_e} \frac{c dt}{R(t)} + \int_{t_e + \Delta t_e}^{t_o} \frac{c dt}{R(t)} = \int_{t_e}^{t_o} \frac{c dt}{R(t)} + \int_{t_o}^{t_o + \Delta t_o} \frac{c dt}{R(t)}$$

$$\frac{\Delta t_e}{R(t_e)} = \frac{\Delta t_o}{R(t_o)} \quad \rightarrow \quad \frac{R(t_o)}{R(t_e)} = \frac{\Delta t_o}{\Delta t_e} = \frac{\lambda_o}{\lambda_e} = 1 + z$$

∴ Observed wavelengths and time intervals appear "stretched" by a factor $x = 1 + z = R_0/R(t)$.

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Fidos and co-moving coordinates

Distance varies in time:

$$D(t)$$

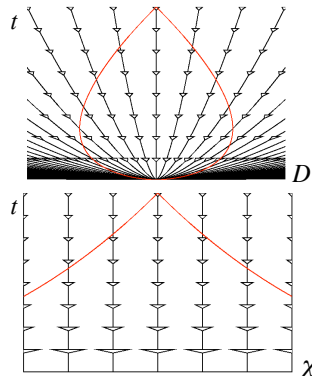
"Fiducial observers" (Fidos)

$$D(t) = R(t) \chi$$

"Co-moving" coordinates

$$\chi \text{ or } D_0 \equiv R_0 \chi$$

Labels the Fidos



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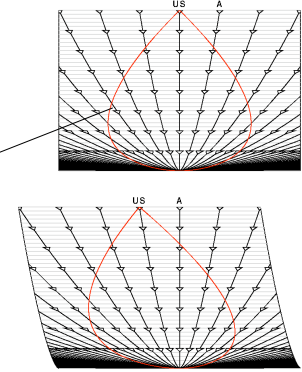
Cosmological Principle

All "co-moving" observers see an equivalent view.

Light ray

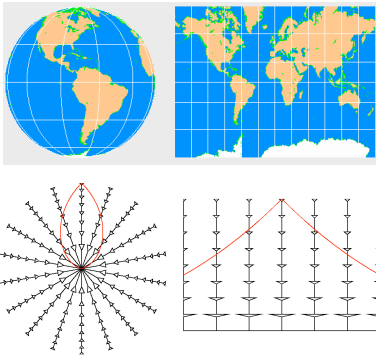
Past light cone:

- looking out = back in time



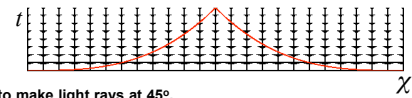
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Coordinate Systems

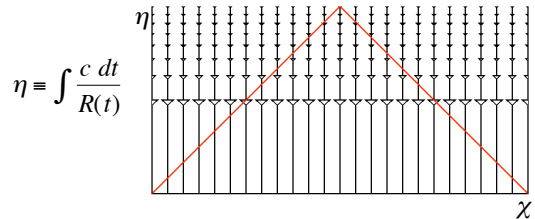


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Conformal time



Stretch time axis to make light rays at 45°.



$$ds^2 = R^2(t) \left(-d\eta^2 + d\chi^2 + S_k^2(\chi) d\psi^2 \right)$$

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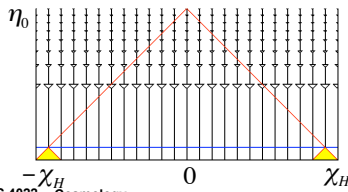
The Horizon: How far can we see?

- We see only a finite patch of the Universe.
- The Horizon may grow with time (if expansion decelerates) or shrink in time (e.g. inflation).

suppose $R(t) = R_0 (t/t_0)^\alpha$ matter - dominated: $\alpha = 2/3$
radiation - dominated: $\alpha = 1/2$

$$\chi_H = \eta_0 = \int_0^{t_0} \frac{c dt}{R(t)} = \frac{c t_0}{(1-\alpha) R_0} \text{ if } \alpha < 1$$

$\Rightarrow \infty$ as $\alpha \rightarrow 1$



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Angular Diameter Distance

- radial distance

- now (when photon received):

$$D_0 = R(t_0) \chi = R_0 \chi$$

- when photon emitted:

$$D_e = R(t_e) \chi = \frac{R(t_e) R_0 \chi}{R_0} = \frac{D_0}{1+z}$$

- angular size

- Fraction of circumference when photon was emitted:

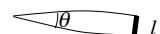
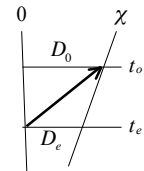
$$\frac{\theta}{2\pi} = \frac{l}{2\pi r(t_e)}$$

- angular diameter distance

$$D_A \equiv \frac{l}{\theta} = r(t_e) = R(t_e) S_k(\chi) = \frac{R(t_e)}{R_0} R_0 S_k(\chi)$$

$$= \frac{R_0 S_k(\chi)}{1+z} = \frac{r(t_0)}{1+z} \equiv \frac{r_0}{1+z}$$

Circumference was smaller by factor $x=1+z$.
Sources look larger/closer.



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Luminosity Distance

- Luminosity (erg s⁻¹) $L = \frac{N h \nu_e}{\Delta t_e}$

- area of photon sphere (when photons observed):

$$A_0 = 4\pi r_0^2 = 4\pi R_0^2 S_k^2(\chi)$$

- redshift:

$$\lambda_0 = \lambda_e (1+z)$$

$$\nu_0 = \nu_e / (1+z)$$

- time dilation: lower photon arrival rate

$$\Delta t_0 = \Delta t_e (1+z)$$

- observed flux (erg cm⁻² s⁻¹)

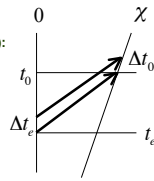
$$F = \frac{N h \nu_0}{A_0 \Delta t_0} = \frac{L}{4\pi r_0^2 (1+z)^2} = \frac{L}{4\pi D_L^2}$$

• Luminosity distance

$$D_L = (1+z)r_0 = (1+z)R_0 S_k(\chi)$$

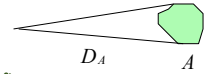
Sources look fainter/farther.

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Surface Brightness

• Solid angle $\Omega = A / D_A^2$



• Surface brightness

- Flux per solid angle (erg s⁻¹ cm⁻² arcsec⁻²)

$$\Sigma \equiv \frac{F}{\Omega} = \frac{L}{4\pi D_L^2} \frac{D_A^2}{A} = \frac{L}{4\pi A (1+z)^4}$$

- decreases very rapidly with z because:
- expansion spreads out the photons
- decreases their energy
- decreases their arrival rate

$$D_L = (1+z)r_0$$

$$D_A = \frac{r_0}{(1+z)}$$

$$r_0 = R_0 S_k(\chi)$$

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Flux Density Spectra

• emitted photons

- (erg s⁻¹ Hz⁻¹)

$$L_\nu(\lambda_e) = \frac{N(\lambda_e) h \nu_e}{\Delta \nu_e \Delta t_e}$$

• redshift

$$\lambda_0 = \lambda_e (1+z)$$

photon numbers

$$N(\lambda_0) = N(\lambda_e)$$

• observed flux density spectra

- (erg cm⁻² s⁻¹ Hz⁻¹)

$$F_\nu(\lambda_0) = \frac{N(\lambda_0) h \nu_0}{\Delta \nu_0 \Delta t_0 A} = \frac{L_\nu(\lambda_e)}{4\pi D_A^2 (1+z)}$$

- (erg cm⁻² s⁻¹ Å⁻¹)

$$F_\lambda(\lambda_0) = \frac{N(\lambda_0) h \nu_0}{\Delta \lambda_0 \Delta t_0 A} = \frac{L_\lambda(\lambda_e)}{4\pi D_A^2 (1+z)^3}$$

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