Lecture 3

Metrics for Curved Geometry

Cosmological Observations in a Curved and Evolving Universe

Non-Euclidian geometries:

(positive / negative curvature)

Evolving geometries:

(expanding / accelerating / decelerating)

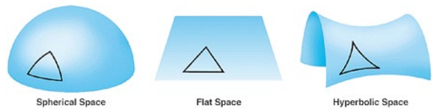
Time-Redshift-Distance relations

Non-Euclidean Geometry

Curved 3-D Spaces

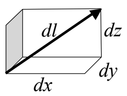
How Does Curvature affect Distance Measurements ?

Is our Universe Curved?



	Closed	Flat	Open
Curvature:	+	0	--
Sum of angles of triangle:	> 180°	= 180°	< 180°
Circumference of circle:	$< 2\pi r$	$= 2\pi r$	$> 2\pi r$
Parallel lines:	converge	remain parallel	diverge
Size:	finite	infinite	infinite
Edge:	no	no	no

Flat Space: Euclidean Geometry



Cartesian coordinates:

1 D: $dl^2 = dx^2$

2 D: $dl^2 = dx^2 + dy^2$

3 D: $dl^2 = dx^2 + dy^2 + dz^2$

4 D: $dl^2 = dw^2 + dx^2 + dy^2 + dz^2$

Metric tensor : coordinates -> distance

$$dl^2 = (\,dx \;\; dy \;\; dz\,)\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}dx\\dy\\dz\end{pmatrix}$$

Summation convention:

$$dl^2 = g_{ij}\,dx^i\,dx^j \equiv \sum_i \sum_j g_{ij}\,dx^i\,dx^j$$

Orthogonal coordinates <--> diagonal metric

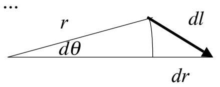
$g_{xx} = g_{yy} = g_{zz} = 1$

$g_{xy} = g_{xz} = g_{yz} = 0$

symmetric: $g_{ij} = g_{ji}$

Polar Coordinates

Radial coordinate r, angles $\phi, \theta, \alpha, \ldots$



1 D: $dl^2 = dr^2$

2 D: $dl^2 = dr^2 + r^2\,d\theta^2$

3 D: $dl^2 = dr^2 + r^2\left(d\theta^2 + \sin^2\theta\;d\phi^2\right)$

4 D: $dl^2 = dr^2 + r^2\left[d\theta^2 + \sin^2\theta\left(d\phi^2 + \sin^2\phi\;d\alpha^2\right)\right]$

$dl^2 = dr^2 + r^2\,d\psi^2$ generic angle: $d\psi^2 = d\theta^2 + \sin^2\theta\,d\phi^2 + \ldots$

$$dl^2 = (\,dr \;\; d\theta \;\; d\phi\,)\begin{pmatrix}1&0&0\\0&r^2&0\\0&0&r^2\sin^2\theta\end{pmatrix}\begin{pmatrix}dr\\d\theta\\d\phi\end{pmatrix}$$

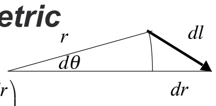
$g_{rr} = ?$ $g_{r\theta} = ?$

$g_{\theta\theta} = ?$

$g_{\phi\phi} = ?$

$g_{\alpha\alpha} = ?$

Using the Metric



$$dl^2 = dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

$$dl^2 = \begin{pmatrix} dr & d\theta & d\phi \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix} \begin{pmatrix} dr \\ d\theta \\ d\phi \end{pmatrix}$$

$$dl_r = \sqrt{g_{rr}}\,dr = dr, \qquad dl_\theta = \sqrt{g_{\theta\theta}}\,d\theta = r\,d\theta, \qquad dl_\phi = ?$$

Radial Distance: $D \equiv \int dl_r = \int_0^r \sqrt{g_{rr}}\,dr = \int_0^r dr = r$

Circumference: $C \equiv \oint dl_\theta = \int_0^{2\pi} \sqrt{g_{\theta\theta}}\,d\theta = \int_0^{2\pi} r\,d\theta = 2\pi r$

Area: $A = \int dA_{r\theta} = \int dl_r\,dl_\theta = \int_0^r \int_0^{2\pi} \sqrt{g_{rr}}\,dr \sqrt{g_{\theta\theta}}\,d\theta = \int_0^r dr \int_0^{2\pi} r\,d\theta = \pi r^2$

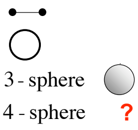
Note: $\int dx\,dy = \int r\,dr\,d\theta$

Same result using metric for any choice of coordinates.

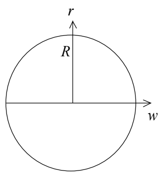
Embedded Spheres

R = radius of curvature

1 − D: $R^2 = x^2$	0 - D 2 points	
2 − D: $R^2 = x^2 + y^2$	1 - D circle	
3 − D: $R^2 = x^2 + y^2 + z^2$	2 - D surface of 3 - sphere	
4 − D: $R^2 = x^2 + y^2 + z^2 + w^2$	3 - D surface of 4 - sphere	?



Metric for 3-D surface of 4-D sphere



4 − sphere: $R^2 = x^2 + y^2 + z^2 + w^2$

i.e. $R^2 = r^2 + w^2$ with $r^2 \equiv x^2 + y^2 + z^2$.

$$0 = 2r\,dr + 2w\,dw \quad \rightarrow \quad dw^2 = \left(\frac{r\,dr}{w}\right)^2 = \frac{r^2\,dr^2}{R^2 - r^2}$$

$dl^2 = dw^2 + dr^2 + r^2 d\psi^2$ 4 - space metric

$= \dfrac{r^2\,dr^2}{R^2 - r^2} + dr^2 + r^2 d\psi^2$ confined to $R^2 = r^2 + w^2$

$$dl^2 = \frac{dr^2}{1 - (r/R)^2} + r^2 d\psi^2 \qquad d\psi^2 = d\theta^2 + \sin^2\theta\, d\phi$$

Metric for a 3 - D space with constant curvature radius R

Non-Euclidean Metrics

$k = -1, 0, +1$ (open, flat, closed)

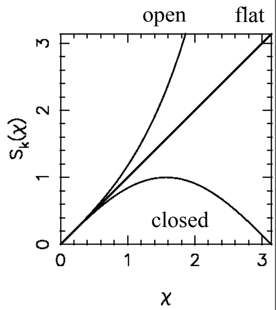
$$dl^2 = \frac{dr^2}{1 - k\,(r/R)^2} + r^2 d\psi^2$$

dimensionless radial coordinates:

$$u = r/R = S_k(\chi)$$

$$dl^2 = R^2\left(\frac{du^2}{1 - k\,u^2} + u^2 d\psi^2\right)$$

$$= R^2\left(d\chi^2 + S_k^{\,2}(\chi)\,d\psi^2\right)$$

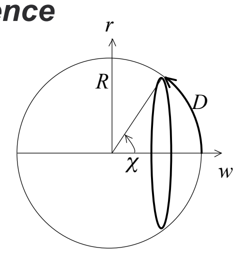


$S_{-1}(\chi) \equiv \sinh(\chi)$, $S_0(\chi) \equiv \chi$, $S_{+1}(\chi) \equiv \sin(\chi)$

Circumference

metric:

$$dl^2 = \frac{dr^2}{1 - k\,(r/R)^2} + r^2\,d\theta^2$$



radial distance (for $k = +1$):

$$D = \int_0^r \frac{dr}{\sqrt{1 - k\,(r/R)^2}} = R\sin^{-1}(r/R)$$

circumference :

$$C = \int_0^{2\pi} r\,d\theta = 2\pi r$$

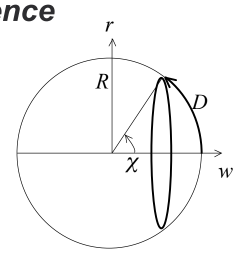
"circumferencial" distance: $r \equiv \dfrac{C}{2\pi} = R\,S_k(D/R) = R\,S_k(\chi)$

If $k = +1$, coordinate r breaks down for $r > R$

Circumference

metric:

$$dl^2 = R^2\left(d\chi^2 + S_k^2(\chi)\,d\theta^2\right)$$



radial distance :

$$D = \int \sqrt{g_{\chi\chi}}\,d\chi = \int_0^\chi R\,d\chi = R\chi$$

circumference :

$$C = \oint \sqrt{g_{\theta\theta}}\,d\theta = \int_0^{2\pi} R\,S_k(\chi)\,d\theta = 2\pi\,R\,S_k(\chi)$$

$$= 2\pi\,D\,\frac{S_k(\chi)}{\chi}$$

Same result for any choice of coordinates.

Angular Diameter

metric :

$$dl^2 = R^2 \left(d\chi^2 + S_k^2(\chi)\, d\theta^2 \right)$$

radial distance :

$$D = \int \sqrt{g_{\chi\chi}}\, d\chi = \int_0^{\chi} R\, d\chi = R\,\chi$$

linear size : ($l << D$)

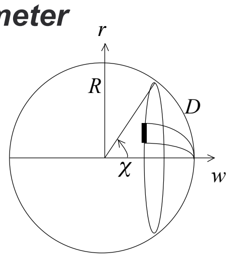
$$l = \int \sqrt{g_{\theta\theta}}\, d\theta = R\, S_k(\chi)\, \theta$$

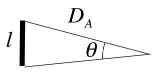
angular size :

$$\theta = \frac{l}{D_A}$$

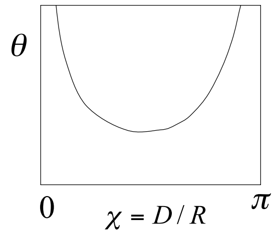
$D = R\ \chi =$ Radial Distance

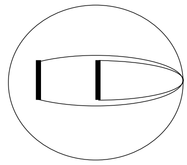
$D_A = R\, S_k(\chi) =$ Angular Diameter Distance





Angular Diameter





$$\theta = \frac{l}{D_A} \qquad D_A = R\, S_k(\chi) = \text{ Angular Diameter Distance}$$

Positive curvature makes objects look larger, hence closer.

Area of Spherical Shell

radial coordinate χ, angles θ, ϕ :

$$dl^2 = R^2 \left[d\chi^2 + S_k^2(\chi) \left(d\theta^2 + \sin^2\theta\ d\phi^2 \right) \right]$$

area of shell :

$$A = \int \sqrt{g_{\theta\theta}}\, d\theta \sqrt{g_{\phi\phi}}\, d\phi$$

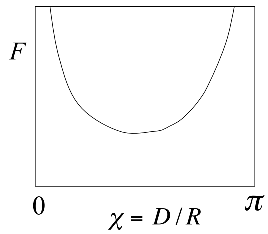
$$= R^2\ S_k^2(\chi) \int_0^{\pi} d\theta\ \sin\theta \int_0^{2\pi} d\phi$$

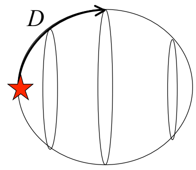
$$= 4\pi\ R^2\ S_k^2(\chi)$$

flux :

$$F = \frac{L}{A} = \frac{L}{4\pi\ D_L^2} \qquad D_L = R\, S_k(\chi) = \text{ Luminosity Distance}$$

Fluxes





$$F = \frac{L}{A} = \frac{L}{4\pi\ D_L^2} \qquad D_L = R\, S_k(\chi) = \text{ Luminosity Distance}$$

Positive curvature makes sources look brighter, hence closer.

Note : $D_L = D_A$ if $R = \text{const.}$

Summary

- The **metric** converts coordinate steps to physical lengths.
- Use the metric to compute lengths, areas, volumes, …
- Radial distance: $D \equiv \int \sqrt{g_{rr}}\, dr = R\,\chi$
- "Circumferencial" distance

$$r \equiv \frac{C}{2\pi} = \left(\frac{A}{4\pi} \right)^{1/2} = \frac{1}{2\pi} \int_0^{2\pi} \sqrt{g_{\phi\phi}}\, d\phi = R\, S_k(\chi) = R\, S_k(D/R)$$

- "Observable" distances, defined in terms of local observables (angles, fluxes), give r, not D.

$$D_A \equiv \frac{l}{\theta} = r \qquad D_L \equiv \left(\frac{L}{4\pi F} \right)^{1/2} = r$$

- r can be smaller than D (positive curvature) or larger (negative curvature) or the same (flat).