Radial Potential

energy and angular momentum :

$$e \equiv \frac{E}{m} = \frac{1}{2}\left[\dot{r}^2 + r^2 \dot{q}^2\right] - \frac{G\,M}{r}$$

$$L \equiv \frac{J}{m} = r^2 \dot{q}$$

effective potential : $e = \frac{\dot{r}^2}{2} + \Phi(r)$

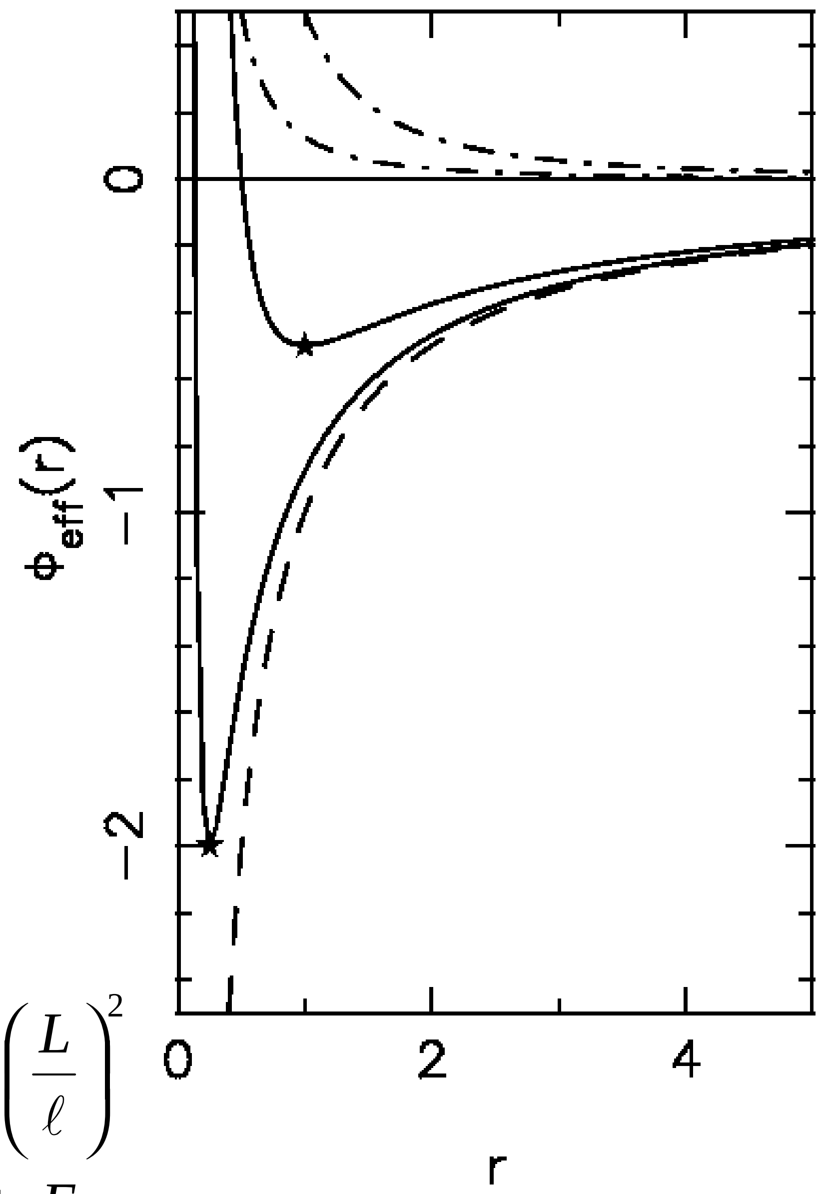
$$\Phi(r) = \frac{L^2}{2\,r^2} - \frac{G\,M}{r} = e_o \left(\frac{\ell^2}{r^2} - 2\,\frac{\ell}{r} \right)$$

circular orbit :

$$\frac{\partial \Phi}{\partial r} = \frac{G\,M}{r^2} - \frac{L^2}{r^3} = 0 \rightarrow r = \frac{L^2}{G\,M} \equiv \ell$$

$$\Phi(\ell) = -e_o \quad e_o \equiv \frac{G\,M}{2\,\ell} = \frac{1}{2}\left(\frac{G\,M}{L}\right)^2 = \frac{1}{2}\left(\frac{L}{\ell}\right)^2$$

Note : ℓ, e_o depend on M and J but not E.



Types of Orbits

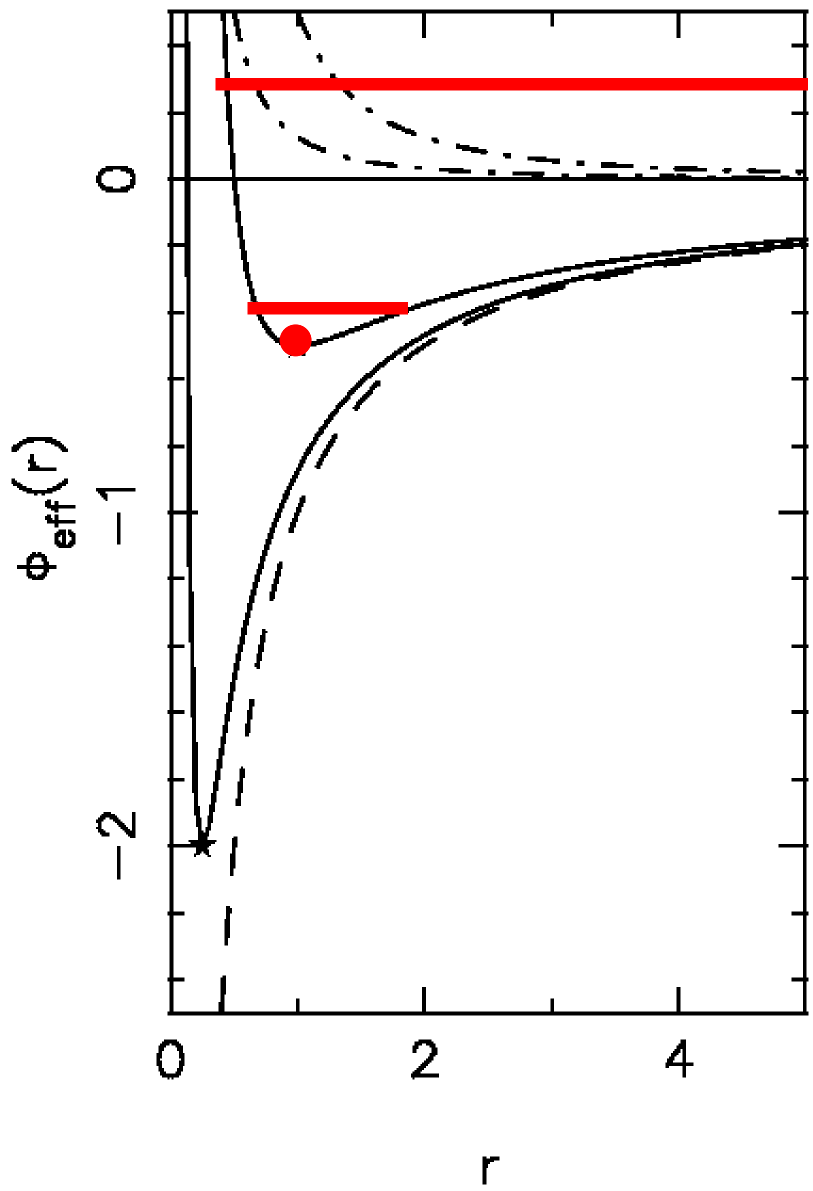
Fix *L*, *E* = min circular

***E* < *0* bound (ellipse)**

***E* > *0* unbound (hyperbola)**

Fix *E* < *0*, *L* = max circular

***L* = min radial**



Turning Points

energy and angular momentum :

$$\boldsymbol{e} = \frac{\dot{r}^2}{2} + \frac{L^2}{2\,r^2} - \frac{G\,M}{r}$$

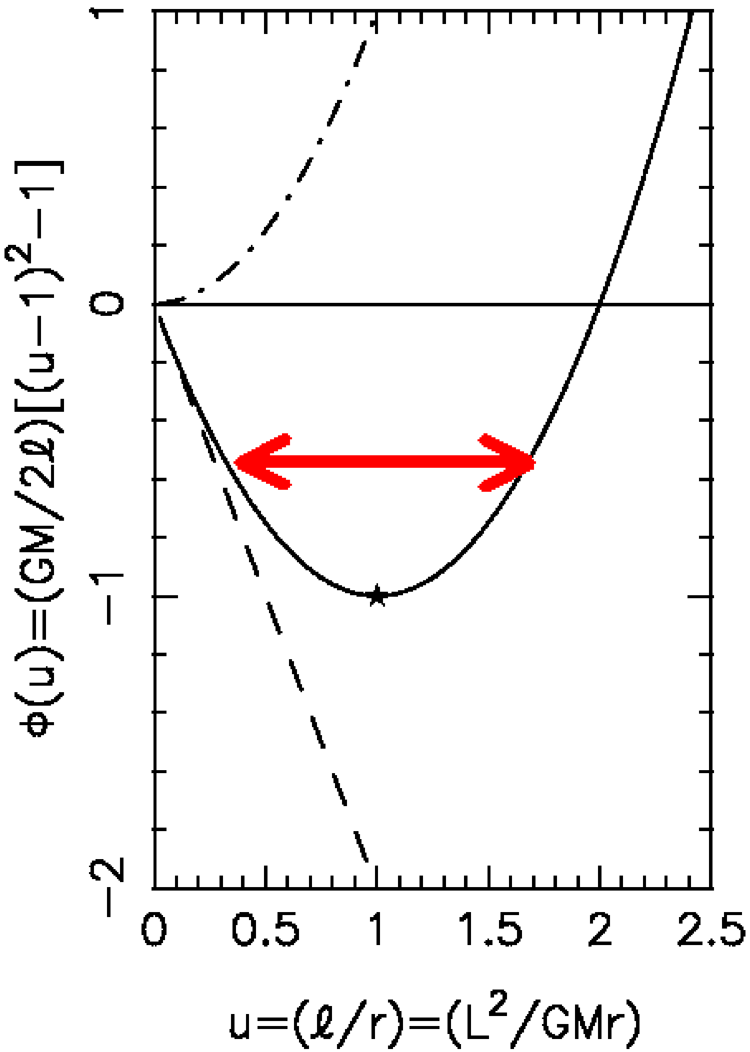
$$= \frac{\dot{r}^2}{2} + \boldsymbol{e}_o \left[\left(u - 1 \right)^2 - 1 \right]$$

$$u \equiv \frac{\ell}{r}$$

turning points :

$$\dot{r}^2 = 0 \rightarrow \left(u - 1 \right)^2 = 1 + \frac{\boldsymbol{e}}{\boldsymbol{e}_o} \equiv e^2$$

$$u^{\pm} = \frac{\ell}{r^{\mp}} = 1 \pm \left(1 + \frac{\boldsymbol{e}}{\boldsymbol{e}_0} \right)^{1/2}$$



Orbit Shape

$$e = \frac{\dot{r}^2}{2} + e_0\left(x^2 - 1\right)$$

$$\dot{r}^2 = 2\,e_0\left(\frac{e}{e_0} + 1 - x^2\right) = \left(\frac{L}{\ell}\right)^2\left(e^2 - x^2\right)$$

$$\frac{dx}{dq} = \frac{\dot{x}}{\dot{q}} = \left(-\frac{\ell\,\dot{r}}{r^2}\right)\Big/\left(\frac{L}{r^2}\right) = -\frac{\ell}{L}\,\dot{r}$$

$$\left(\frac{dx}{dq}\right)^2 + x^2 = e^2$$

simple harmonic oscillator !

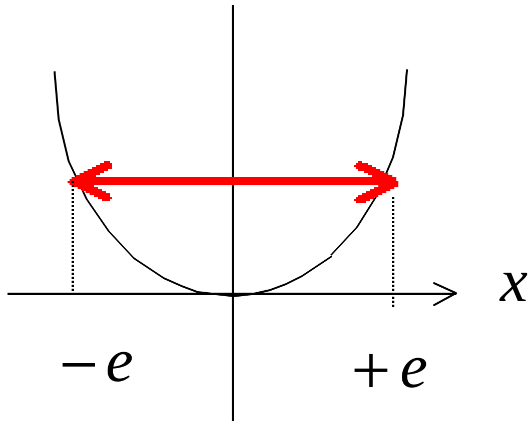
$$x = e\cos q \quad \rightarrow \quad \boxed{r = \frac{\ell}{1 + e\cos q}}$$

$$x \equiv \frac{\ell}{r} - 1 \qquad \dot{x} = -\frac{\ell\,\dot{r}}{r^2}$$

$$\ell \equiv \frac{L^2}{G\,M} \qquad L \equiv r^2\,\dot{q}$$

$$e_0 \equiv \frac{1}{2}\left(\frac{L}{\ell}\right)^2$$

$$e^2 \equiv \frac{e}{e_0} + 1$$



Conic Sections

$$r = \frac{\ell}{1 + e\cos q}$$

$e =$ eccentricity

> 1 hyperbola

$= 1$ parabola

< 1 ellipse

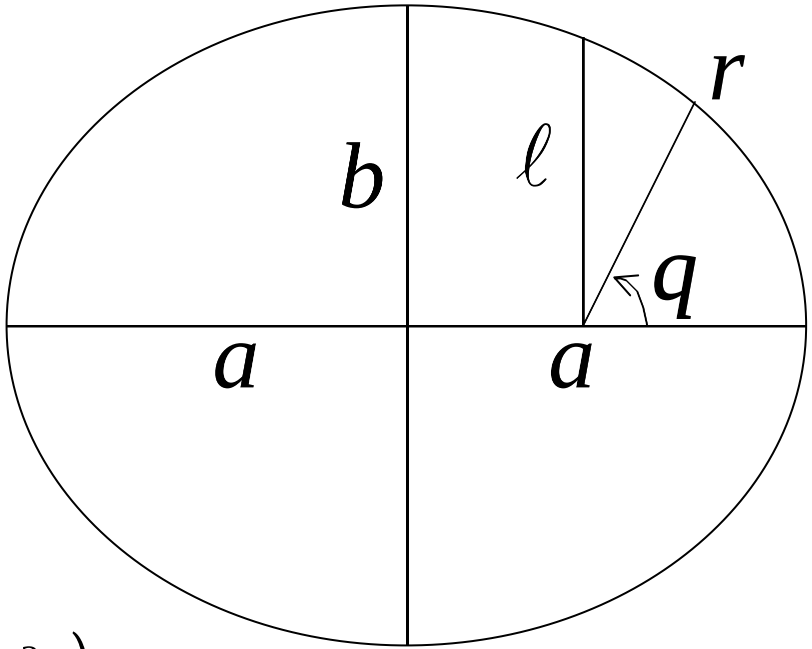
$= 0$ circle

$\ell =$ semi - latus rectum $= a\left(1 - e^2\right)$

$a =$ semi - major axis

$b =$ semi - minor axis

$= a\left(1 - e^2\right)^{1/2}$



$q = 0 \quad r = \frac{\ell}{1+e} = a(1-e)$ periastron

$q = 90^\circ \quad r = \ell = a(1-e^2)$

$q = 180^\circ \quad r = \frac{\ell}{1-e} = a(1+e)$ apastron

Orbital Speed

$$r = \frac{\ell}{1 + e\cos q} \qquad \ell \equiv \frac{L^2}{G\,M} = a\left(1 - e^2\right)$$

$$\boldsymbol{e} = -\frac{G\,M}{2\,\ell}\left(1 - e^2\right) = -\frac{G\,M}{2\,a}$$

$$\text{Energy:}$$

$$V^2 = 2\,\boldsymbol{e} + \frac{2\,G\,M}{r} = G\,M\left[\frac{2}{r} - \frac{1}{a}\right]$$

Motion in Time

$$r = \frac{\ell}{1 + e \cos q} \qquad \ell \equiv \frac{L^2}{G\, M}$$

$$\dot{\grave{e}} = \frac{L}{r^2} = \frac{L}{\ell^2}\,(1 + e \cos q\,)^2$$

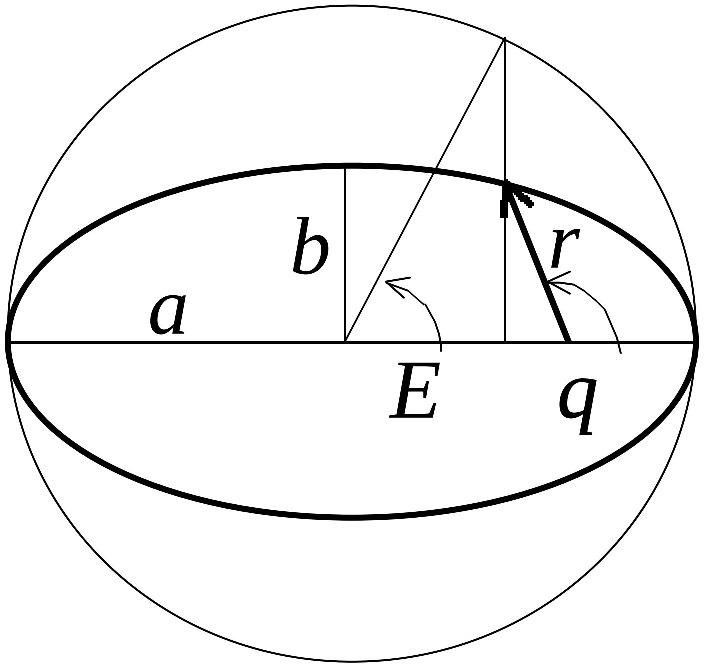
$$\frac{dq}{(1 + e \cos q\,)^2} = \frac{L}{\ell^2}\,dt$$

$$\int \frac{dq}{(1 + e \cos q\,)^2} = \frac{L}{\ell^2}\,(\,t - T\,)$$

No analytic solution for *q(t)*.

Eccentric Anomaly

auxiliary circle



$$b = a\sqrt{1-e^{2}} =$$

q = true anomaly

E = eccentric anomaly

$$x = a\cos E = a\,e + r\cos q$$

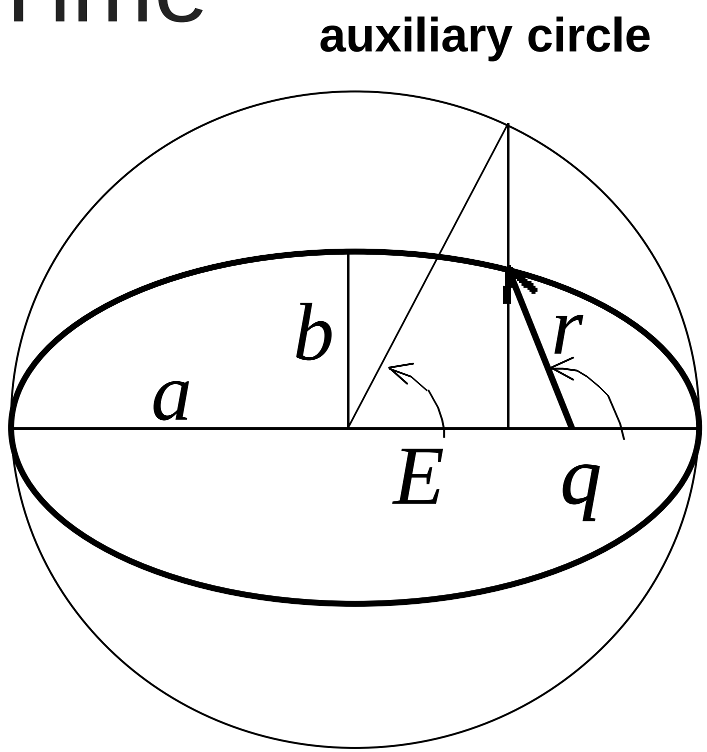
$$y = b\sin E = r\sin q$$

$$r^{2} = a^{2}\,(\cos^{2}E - e)^{2} + a^{2}\,(1-e^{2})\,\sin^{2}E$$

$$r = a\left(1 - e\cos E\right) \rightarrow dr = a\,e\sin E\,dE$$

.......Kepler's equation giving $E(t)$

Motion in Time



h = mean anomaly

f = orbital phase

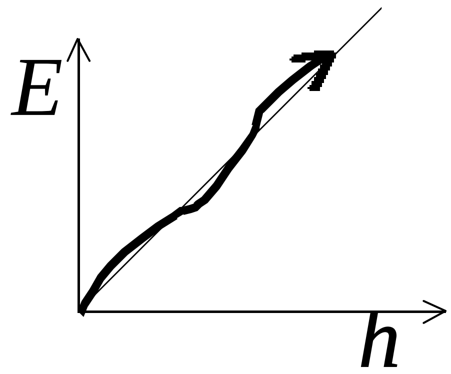
P = orbital period

T = time of periastron passage

Kepler's equation :

$$E - e \sin E = h = 2p\, f = \frac{2p}{P}(t - T)$$

iterate to find *E(t)*



$$\tan\left(\frac{q}{2}\right) = \left(\frac{1+e}{1-e}\right)^{1/2} \tan\left(\frac{E}{2}\right)$$